T-79.159 Cryptography and Data Security

Lecture 5: Public Key Algorithms

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Recap: what we have done

- First lecture: general overview
- Second lecture: secret-key cryptography
- Third lecture: Modes of operation
- Fourth lecture: Hash functions
  Lectures 2–4 are all about secret-key cryptography!
- Today: Public key algorithms

Problems of symmetric model (1/3)

- Alice and Bob need to share a key
  - ⋆ distributed over a private channel
  - ⋆ say, when they meet in a pub
- Private channels are very expensive
  - ⋆ especially in Finland

Problems of symmetric model (2/3)

Huge number of keys when scaling:

- n participants who want to communicate pairwise secretly
- Every pair needs a secret key, there are $\binom{n}{2} = \frac{n^2-n}{2}$ pairs
- Thus, $\frac{n^2-n}{2}$ keys must be pre-distributed!
- Every participant needs to store n different keys
- Say, $n = 6 \cdot 10^9 \ldots$

Problems of symmetric model (2/3)

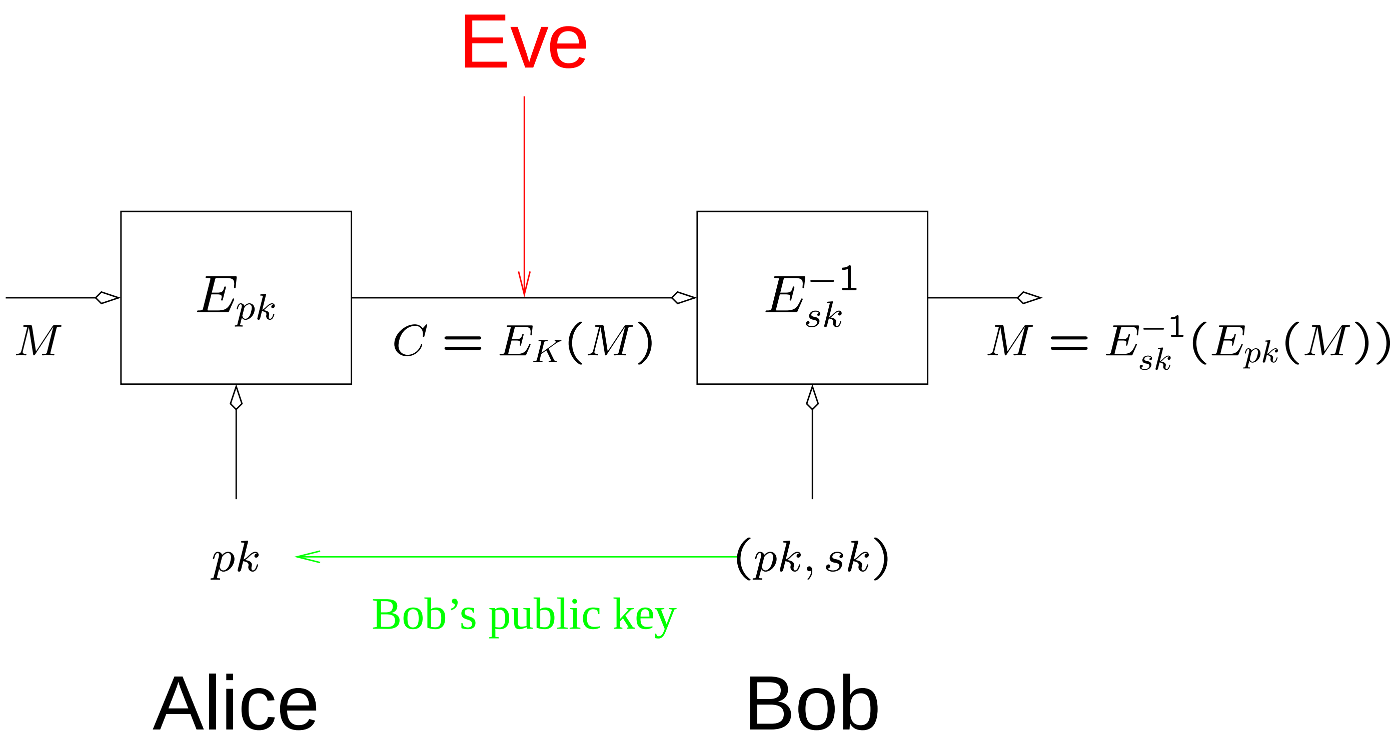
Non-repudiation:

- You can authenticate yourself and your messages to your friends by using MAC=s
- However, MAC-s use shared key
- Therefore, you cannot prove to third parties that messages were really sent by your friend and not by yourself!

Public key cryptography: mysterious helper

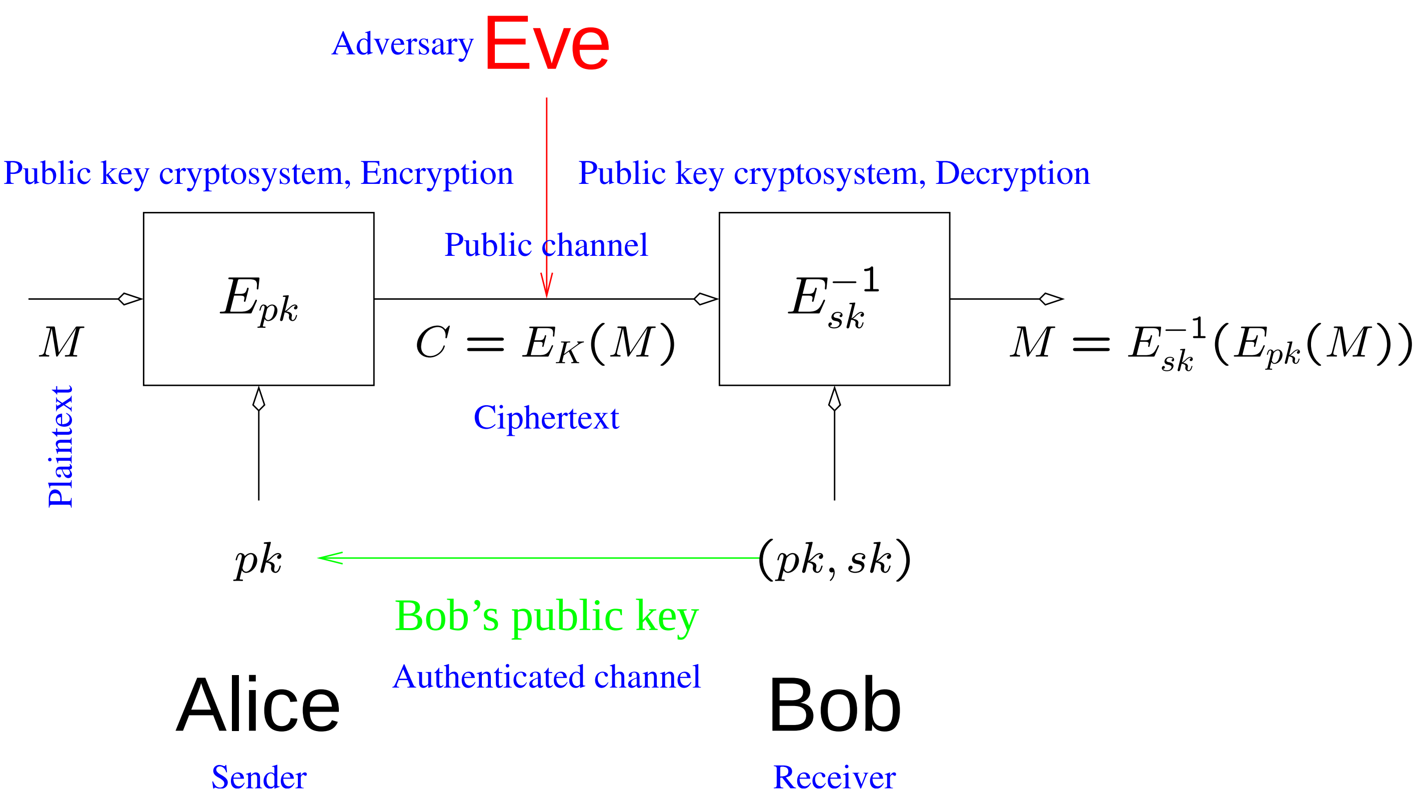
- All mentioned problems can be solved by using PKC
- Basic idea: everybody has a pair (pk, sk) of public and secret keys
- If you want to send to me a message, you first fetch my pk from somewhere (phone book?), then encrypt a message by pk and send the result to me
- I will decrypt the ciphertext by using my secret key

PKC: model



Eve
E_{pk}
E_{sk}^{-1}
M
$C = E_K(M)$
$M = E_{sk}^{-1}(E_{pk}(M))$
pk
(pk, sk)
Bob's public key
Alice
Bob

PKC: model



Alice obtains public key from an *authenticated* channel, no privacy during this is necessary!

Public-Key Cryptography: Assumptions

- PKC bases on clear mathematics
  - ⋆ Existence of one-way functions, and related primitives
- "Crazy" solutions (AES-like or DES-like) are not accepted
- Important to know: PKC bases on the assumption that there is *one* OWF
- If this OWF gets "broken", it can be substituted with another one — assuming that *OWFs exist*

Etude: Elementary mathematics (1/2)

(Known from the discrete mathematics course)

- For any integer n, $\mathbb{Z}_n = \{0, \ldots, n-1\}$
- $\mathbb{Z}_n$ is an additive group: $a + b = c \mod n$. E.g., $7 + 12 = 19 \equiv 6 \mod 13$, thus $7 + 12 = 6$ in $\mathbb{Z}_{13}$
- Analogously, one can define modular multiplication: $7 \cdot 12 = 84 \equiv 6 \mod 13$
- However, $\mathbb{Z}_n$ is not a group w.r.t. multiplication, since not all elements of $\mathbb{Z}_n$ have inverses

Etude: Elementary mathematics (2/2)

(Known from the discrete mathematics course)

- y is inverse of x modulo n iff $xy = 1 \mod n$
- For any integer n, $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n : x$ has an inverse modulo $n\}$
- Elementary result: x has an inverse iff $\mathsf{gcd}(x, n) = 1$
- E.g., $4^{-1} \equiv 10 \mod 13$ since $4 \cdot 10 = 40 \equiv 1 \mod 13$, but 4 does not have an inverse modulo 12, since $\mathsf{gcd}(4, 12) = 4 \neq 1$
- Euler's totient function $\varphi(n) := \sharp\mathbb{Z}_n^*$

RSA (1/2)

- The first proposed cryptosystem (Rivest, Adleman, Shamir, 1977), works in $\mathbb{Z}_n^*$ where $n = pq$ is a product of two secret primes

- Still the most used public-key cryptosystem

– Slow key generation

– Sub-exponential attacks known, thus long keys

– Not readily generalizable to other algebraic structures

– No semantic security

RSA (2/2)

- Key generation: generate two random large primes p, q, set $n = pq$. Choose an e, s.t. $\gcd(e, \varphi(n)) = 1$. Compute $d := e^{-1} \mod \varphi(n)$
- (n, e) is the public key, (p, q, d) is the secret key.
- To encrypt an $x \in \mathbb{Z}_n^*$, compute $y = x^e \mod n$
- To decrypt $y \in \mathbb{Z}_n^*$, compute $y^d \mod n$
- Clearly, $x^{ed \mod \varphi(n)} \equiv x \mod n$

RSA: efficiency

- Usually, $e = 3$ or $e = 2^{16}+1$ is used. This speeds up exponentiation: $x^3 \equiv x^2 \cdot x \mod n$ can be computed in two multiplications, $x^{2^{16}+1} = (((x^2)^2)^{\cdots 2})^2 \cdot x \mod n$ in 17 multiplications. Thus, encryption is fast.

- Decryption needs in average $k/2$ multiplications when k-bit modulus is used. (Can be sped up by using the Chinese Remainder Theorem.)

- Generating primes p and q can be done efficiently by using randomized algorithms (Rabin-Williams, . . .)

See algorithms from the textbook

RSA: security (1/3)

- If n can be factorized then one can recompute $\varphi(n) = (p-1)(q-1)$, and hence also $d = e^{-1} \mod \varphi(n)$
  - ⋆ Factoring is easy $\Rightarrow$ RSA is broken
- Best factorization algorithms: quadratic field sieve, generalized number field sieve, elliptic curve factorization method
- Modulus must be at least 1024-bit long to resist factoring
- It is *not* known whether breaking RSA is equivalent to factoring, it is believed that it is not

RSA: security (2/3)

- RSA security (in the sense of message recovery) bases on the difficulty of computing roots (the RSA problem): given (x, e) and an RSA modulus n, it is difficult to compute $x^{e^{-1}} \mod n$

- *Semantic security*: you can choose x_1 and x_2, and let the black box one of them (as chosen by the black box). You get the ciphertext $y = E_K(m_b)$ for random $b \leftarrow \{0, 1\}$. You must guess the value of b

- Example: you know that Napoleon is either encrypting "Attack" or "Relax". Clearly it is relevant that the encryption scheme must be semantically secure!

RSA: security (3/3)

- RSA is not semantically secure, since it is deterministic: you can encrypt both “Attack” and “Relax” yourself, and compare the outcomes with the received ciphertext

- Various methods exist for making RSA semantically secure; many ad hoc methods have been broken (including PKCS as described in the textbook)

- RSA together with OAEP (Optimal Asymmetric Encryption Padding, Bellare and Rogaway, 1994 — as improved by Shoup and others in 2001) is *provably* semantically secure, but the resulting scheme is quite complex

Alternative: Discrete logarithm problem

- Take any “good” group G
  - $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$
  - Elliptic curves
- In these groups: Exponentiation g^x is easy, but given (g, g^x) it is difficult to find x
  - This is the *discrete logarithm problem*: $(g, g^x) \rightarrow x$

Elliptic curve

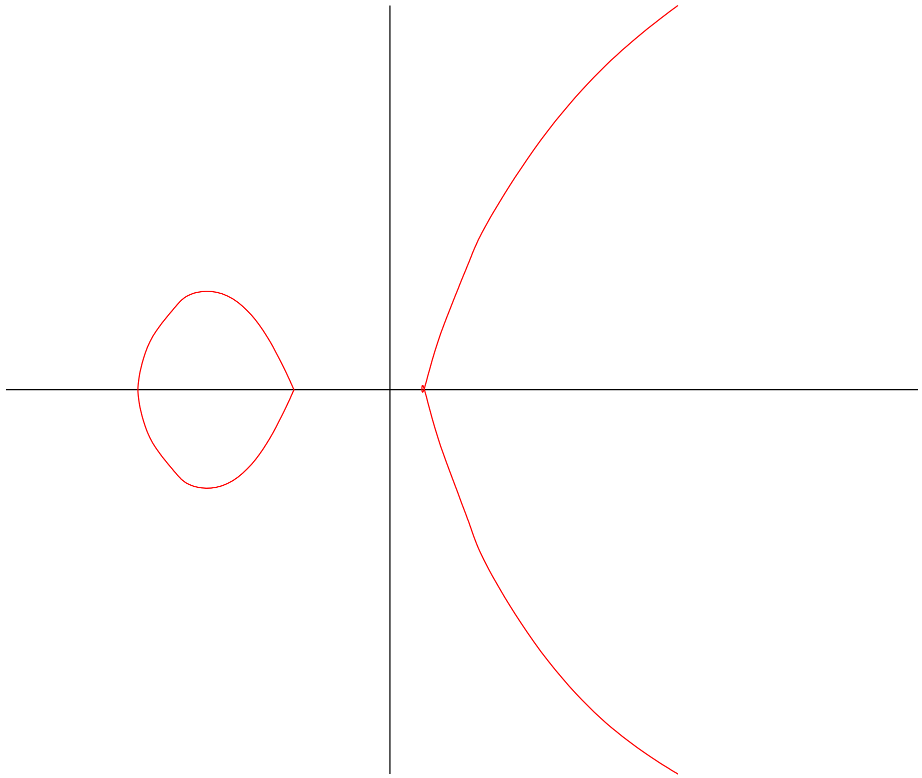
Fix a field $\mathbb{F}$ of characteristic $c \neq 2, 3$ (for those cases, formulas are slightly different). Elliptic curve is a nonsingular cubic curve,

$$C : y^2 = x^3 + ax + b$$

Nonsingular: $x^3 + ax + b$ has no repeated factors

Elliptic curve points: all pairs $(x, y) \in \mathbb{F}^2$ that belong to C together with a special point $\mathcal{O}$ at the infinity.

Elliptic curve: illustration

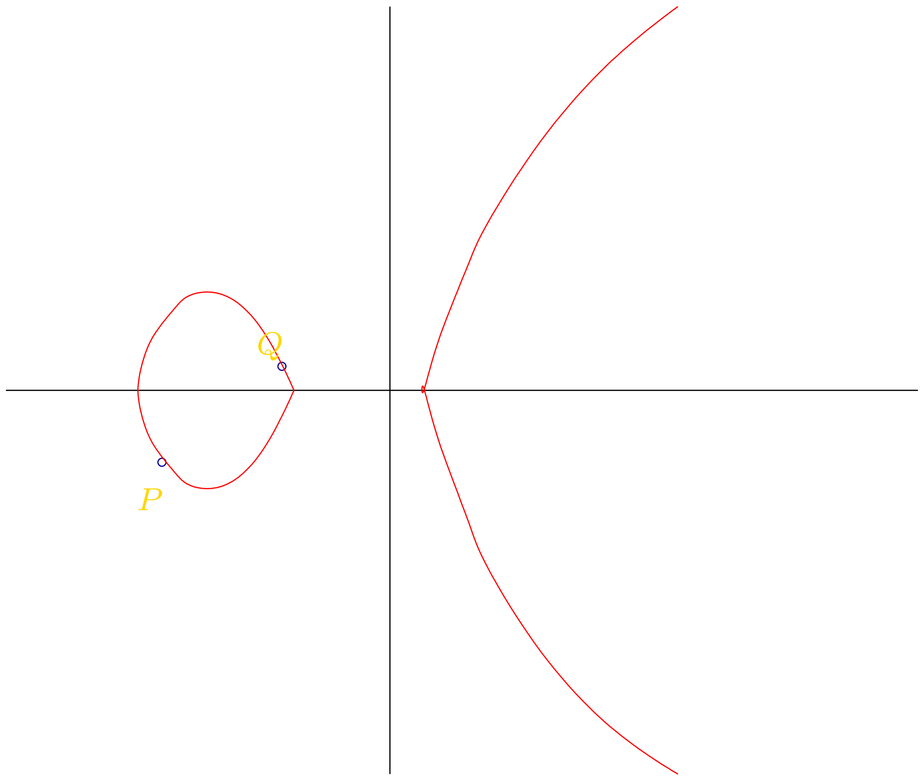


Here, $\mathbb{F} = \mathbb{R}$!

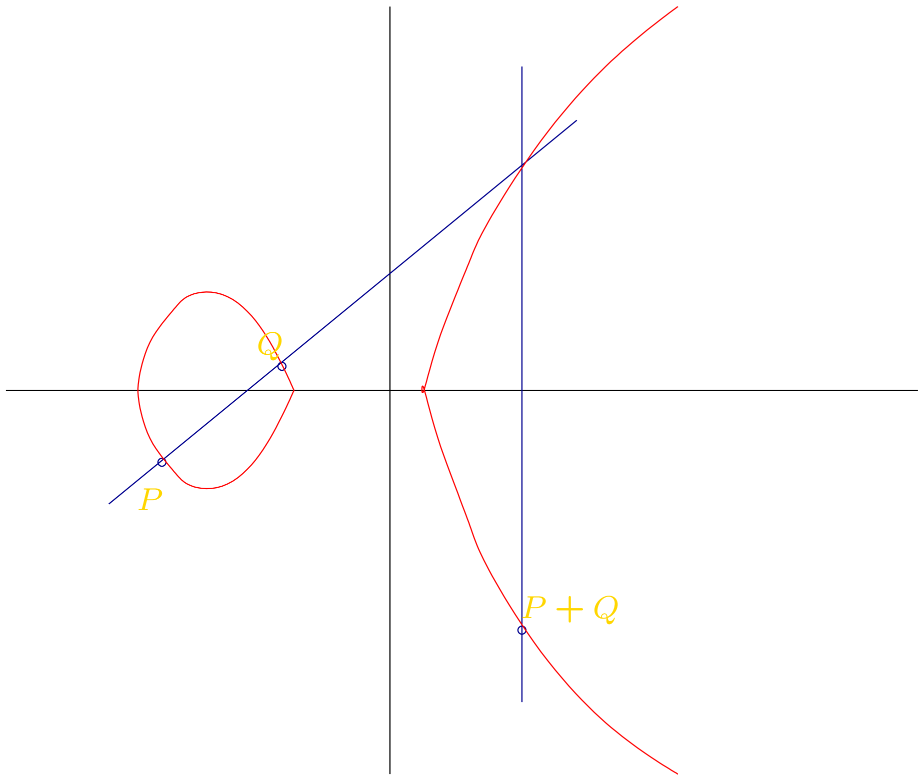
Elliptic curve group

- Take $E(C)$ be the set of all EC points
- For two points P, Q on the curve, define $P + Q$ as follows:
- ...Draw a line that crosses P and Q
- ...Find the third intersection point of this line and the curve
- Mirror this point w.r.t. y-axis

Elliptic curve group: illustration



Elliptic curve group: illustration



EC addition: formulas

Curve: $y^2 = x^3 + ax + b$, $\mathbb{F} = \mathbb{R}$. Define group $E_{\mathbb{F}}(C)$ as follows.

Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$. If $Q = (x_1, -y_1)$, define $P + Q = \mathcal{O}$. Otherwise, define the slope of line connecting P and Q: $\lambda = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1}, & P \neq Q, \\ \frac{3x_1^2 + a}{2y_1}, & P = Q. \end{cases}$

Then $P + Q = (x_3, y_3) = (\lambda^2 - x_1 - x_2, \lambda(x_1 - x_3) - y_1)$.

Special cases when one of the two addends is $\mathcal{O}$: $P + \mathcal{O} = \mathcal{O} + P = P$.

EC group

Theorem Let $\mathbb{F}$ be an *arbitrary* field of characteristic $c \neq 2, 3$. Let $C : y^2 = x^3 + ax + b$. Then $(E_{\mathbb{F}}(C), +)$ is a group w.r.t. addition defined in previous slide.

Unit element: $\mathcal{O}$

Inverse: $-\mathcal{O} = \mathcal{O}$, $-(x, y) = (x, -y)$

Commutativity: easy

Associativity: harder to prove

Discrete logarithm problem in EC group

- Fix the field $\mathbb{F} = \mathrm{GF}(q)$, usually $q = 2^p$ or $q = p$ for a prime p, and $q \geq 2^{160}$

- Given $g \in E_{\mathbb{F}}(C)$ of large order, and a random $x \in \mathbb{Z}_{\mathrm{ord}\, g}$, compute x from (g, xg)

- Note: here we use the additive notation. (xg is exponentiation!)

- Believed to be hard: the best algorithm to solve the discrete logarithm problem on a random curve takes $\approx \sqrt{q}$ steps

Algorithms for discrete logarithm problem

Generic algorithms (work for all groups, do not use the structure of group):

- Exhaustive search
- Shanks's baby-step giant-step
- Pollard's rho algorithm
- Pohlig-Hellman algorithm

Algorithms for discrete logarithm problem

Tailored algorithm (for specific groups):

- Index calculus for DL problem in $\mathbb{Z}_p^*$
- DL in $(\mathbb{Z}_p, +)$ can be solved trivially!
- No tailored algorithms are known for randomly chosen elliptic curves!

DLP: Exhaustive search

Given (g, h), $h = g^x$ for unknown x:

- Successively compute g^0, g^1, g^2, $\ldots$, until h is obtained
- Requires 1 multiplication per step, hence x multiplications in total
- Asymptotically: $O(\mathrm{ord}\, g)$ multiplications, $\mathrm{ord}\, g$ is the order of g

For function f, $g = O(f)$ if for some constant c, $g(x) \leq cf(x)$ for all x

Recommendations for a good group

For the best algorithm for DL to take $\geq 2^k$ steps:

- To dwarf the rho algorithm, choose $n \geq 2k$
- To dwarf the Pohlig-Hellman algorithm, make sure that the greatest divisor p of $\operatorname{ord} g$ is big, $p \geq 2k$. Usually, g is chosen to generate a subgroup of prime order
- Choose a group without any tailored algorithms for DL

A randomly chosen EC group over $\mathrm{GF}(q)$, $q = 2^p$ or $q = p$, with $q \geq 2^{160}$ seems to be secure

Diffie-Hellman key exchange

Assume we have a fixed group G and an $g \in G$ with large order

Alice Bob

Private input: x_A (secret key)

Private input: x_B (secret key)

Compute $y_A := g^{x_A}$

$\xrightarrow{y_A}$

Compute $y_B := g^{x_B}$

$\xleftarrow{y_B}$

Output: $y_B^{x_A} = g^{x_A x_B}$

Output: $y_A^{x_B} = g^{x_A x_B}$

Alternatively, y_A is Alice's public key, y_B is Bob's public key, and both can be fetched from a directory

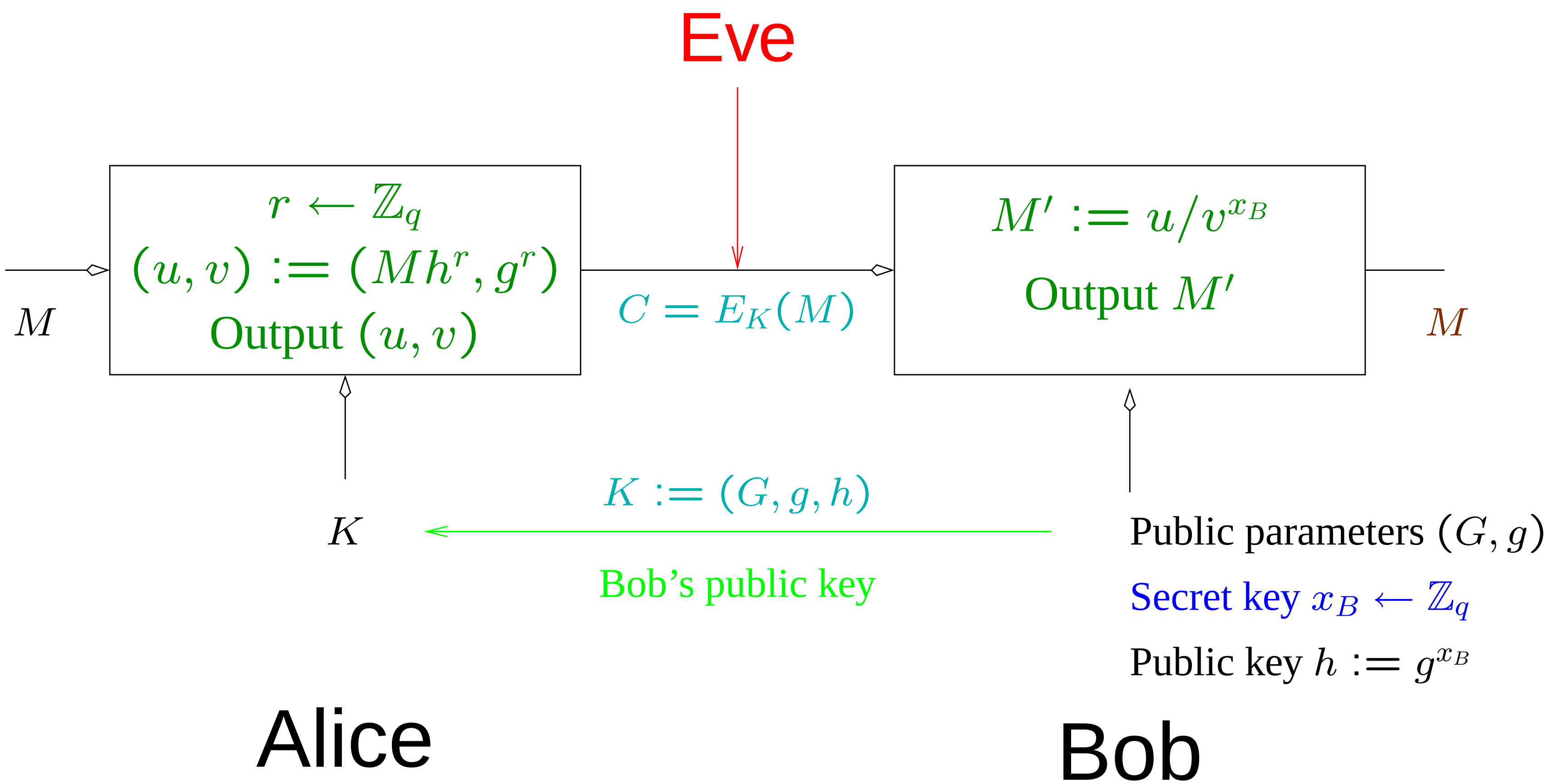
Security of the DH key exchange

Adversary is successful, if, given (g, g^{x_A}, g^{x_B}) she can compute $g^{x_A x_B}$. This is called the *Diffie-Hellman problem* (DH problem).

If DL problem is tractable, then so is the DH problem: compute x_A from (g, g^{x_A}) and then compute $g^{x_A x_B}$ from (g, x_A, g^{x_B})

It is *not* known, if the opposite reduction holds, but the best known algorithms for the DH problem need solving the DL problem

ElGamal cryptosystem



Security of the ElGamal cryptosystem

Message recovery from (mh^r, g^r) and public key $h = g^x$: can be done if DH is tractable. (Compute $h^r = g^{xr}$ from g^r and g^x.)

Is the opposite true? (Can one solve DH, if it is feasible to recover m from (mh^r, g^r) and $h = g^x$?)

Yes, since then one can also recover $h^r = g^{rx}$.

Thus: one can use any group where the DH problem is hard

ECC: ElGamal over an elliptic curve group

More stringent security notions

- *Semantic security*: given m_0 and m_1, distinguish random encryptions of m_0 from m_1

- ... E.g., was the plaintext "yes" or "now"?

- Equivalent (informal) definition: given ciphertext of unknown plaintext m, decide where $P(m)$ is true for some predicate P

- ... E.g., decide whether plaintext contained the word "attack"

Semantic Security of ElGamal

- **Theorem (Jakobsson, Tsiounis, Yung, 1998).** ElGamal is semantically secure if the following *Decisional Diffie-Hellman* (DDH) problem is hard: Given (g, g^x, g^y, h), decide whether $h = g^{xy}$ or $h = g^z$ for random z.

- ElGamal is not secure against the chosen ciphertext attack. Why? (Try to solve)

  ⋆ (Hint: use the *homomorphic* property $E_K(m_1 + m_2) = E_K(m_1)E_K(m_2)$.)

PKC: brief overview

- ECC: ElGamal over EC. Short keys (≥ 160 bits), fast key generation. Semantically secure. Can be made secure against the CCA. Security bases on the DDH assumption in elliptic curves

- RSA. Long keys (≥ 1024 bits), slow key generation, fast encryption. Can be made semantically secure by using the OAEP. Security bases on the RSA assumption

- Other systems: NTRU (long keys, ≥ 1700 bits, $100 \dots 300$ times faster than RSA, less known and studied), XTR (a variant of ElGamal in $\mathrm{GF}(p^6)$, key ≥ 340 bits, approximately as fast as ECC, security bases on the DDH assumption in $\mathbb{Z}_p^*$), ...

Next time

- Identification
- Digital signatures
- Zero-knowledge