T-79.149 Discrete Structures, Autumn 2004
Tutorial 7, 3 November

1. Verify the following properties of the Riemann–Stieltjes -integral, assuming that the integrals appearing in the formulas are well defined:

   (a) Linearity:

$$\begin{aligned}\int_a^b (c_1 f_1 + c_2 f_2)\, dg &= c_1 \int_a^b f_1\, dg + c_2 \int_a^b f_2\, dg, \\ \int_a^b f\, d(c_1 g_1 + c_2 g_2) &= c_1 \int_a^b f\, dg_1 + c_2 \int_a^b f\, dg_2.\end{aligned}$$

   (b) Reduction to Riemann integral: for a continuously differentiable function g,

$$\int_a^b f(t)\, dg(t) = \int_a^b f(t) g'(t)\, dt.$$

2. Let f and g be continuous functions and $a, b \in \mathbf{Z}$. Verify the correctness of the following formulas, assuming that the integrals contained in them are well defined:

$$\begin{aligned}\int_a^b f(t)\, dg(\lceil t \rceil) &= \sum_{a \le k < b} f(k) \Delta g(k), \qquad \Delta g(k) = g(k+1) - g(k); \\ \int_a^b f(\lceil t \rceil)\, dg(t) &= \sum_{a < k \le b} f(k) \nabla g(k), \qquad \nabla g(k) = g(k) - g(k-1).\end{aligned}$$

   Derive from the preceding formulas the following "partial summation rule":

$$\sum_{a \le k < b} f(k) \Delta g(k) = \Big/_a^b f(k) g(k) - \sum_{a < k \le b} g(k) \nabla f(k).$$

3. Use Euler's summation formula to estimate the following sums:

   (a) Sum $\sum_{1 \le k < n} k^{1/2}$ up to order $O(1)$.

   (b) Sum $\sum_{1 \le k < n} k^r$, $r \in \mathbf{N}$, exactly.