T-79.149 Discrete Structures, Autumn 2004

Tutorial 4, 13 October

1. The "multipower" $\mathcal{M}(\mathcal{A}) = (C, w_C)$ of a weighted combinatorial family $\mathcal{A} = (A, w_A)$ is defined as follows. The ambient set C of the multipower consists of all the "multisets" over A, $\{\alpha_1^{j_1}, \ldots, \alpha_k^{j_k}\}$, where $\alpha_i \in A$ for each i, and superscript j_i indicates the order (number of occurrences) of element α_i within the multiset. The weight function for the structures in C is defined as:

$$w_C(\{\alpha_1^{j_1}, \ldots, \alpha_k^{j_k}\}) = j_1 w_A(\alpha_1) + \cdots + j_k w_A(\alpha_k).$$

Prove that this construction is ogf-admissible, with the corresponding ogf operator being:

$$c(z) = \exp(a(z) + \frac{1}{2}a(z^2) + \frac{1}{3}a(z^3) + \cdots).$$

(*Hint:* Observe that $\mathcal{M}(\mathcal{A}) \tilde{\to} \prod_{\alpha \in A} \{\alpha\}^*$.)

2. Use the method of combinatorial constructions ("the operator method") to determine the following ordinary generating functions:

   (a) The number of n-element subsets of a given m-element set, using the powerset construction. (*Hint:* Consider first the ogf of a given m-element set. How many structures does it contain? What is their weight distribution?)

   (b) The number of n-element "multisubsets" (subsets with repetition) of a given m-element set, using the multipower construction from Problem 1.

3. Consider the placement of n identical balls in k distinguishable bins, i.e. the ordered k-*compositions* of the number n: $n = n_1 + \cdots + n_k$. Determine the ogf of the k-compositions of n for a fixed value of k, and the number of k-compositions where: (a) $n_i \geq i$ for all $i = 1, \ldots, k$, (b) n and all the n_i's are even, (c) all the n_i's are odd. (*Hint:* Each component n_i of a given k-composition can be thought of as a sequence of n_i "balls" or "ones".)