1. The “multipower” $M(A) = (C, w_C)$ of a weighted combinatorial family $A = (A, w_A)$ is defined as follows. The ambient set $C$ of the multipower consists of all the “multisets” over $A$, $\{\alpha_1^{j_1}, \ldots, \alpha_k^{j_k}\}$, where $\alpha_i \in A$ for each $i$, and superscript $j_i$ indicates the order (number of occurrences) of element $\alpha_i$ within the multiset. The weight function for the structures in $C$ is defined as:

$$w_C(\{\alpha_1^{j_1}, \ldots, \alpha_k^{j_k}\}) = j_1 w_A(\alpha_1) + \cdots + j_k w_A(\alpha_k).$$

Prove that this construction is ogf-admissible, with the corresponding ogf operator being:

$$c(z) = \exp(a(z) + \frac{1}{2}a(z^2) + \frac{1}{3}a(z^3) + \cdots).$$

(Hint: Observe that $M(A) \sim \prod_{\alpha \in A} \{\alpha\}^*$.)

2. Use the method of combinatorial constructions (“the operator method”) to determine the following ordinary generating functions:

(a) The number of $n$-element subsets of a given $m$-element set, using the powerset construction. (Hint: Consider first the ogf of a given $m$-element set. How many structures does it contain? What is their weight distribution?)

(b) The number of $n$-element “multisubsets” (subsets with repetition) of a given $m$-element set, using the multipower construction from Problem 1.

3. Consider the placement of $n$ identical balls in $k$ distinguishable bins, i.e. the ordered $k$-compositions of the number $n$: $n = n_1 + \cdots + n_k$. Determine the ogf of the $k$-compositions of $n$ for a fixed value of $k$, and the number of $k$-compositions where: (a) $n_i \geq i$ for all $i = 1, \ldots, k$, (b) $n$ and all the $n_i$’s are even, (c) all the $n_i$’s are odd. (Hint: Each component $n_i$ of a given $k$-composition can be thought of as a sequence of $n_i$ “balls” or “ones”.)