1. Prove that if \( a = \langle a_n \rangle \), \( b = \langle b_n \rangle \) and \( c = \langle c_n \rangle \) are sequences (of complex numbers) satisfying \( c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \), then their exponential generating functions satisfy \( \hat{c}(z) = \hat{a}(z) \hat{b}(z) \).

2. The Bell number \( b_n \) indicates how many different partitions (equivalence relations) can be constructed on a set of \( n \) elements. (In terms of Stirling numbers of the 2nd kind one could thus write \( b_n = \sum_{k=1}^{n} \left\{\binom{n}{k}\right\} \).) Show that the Bell numbers \( b_n \) satisfy the recurrence

\[
b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k, \quad b_0 = 1,
\]

and derive from this the exponential generating function \( \hat{b}(z) \) for the sequence \( b = \langle b_n \rangle \). (Hint: Differentiate the series defining the sequence’s egf and solve the resulting differential equation.)

3. Let \( A = (A, w_A) \) and \( B = (B, w_B) \) be two weighted families of combinatorial structures. Construct their product \( A \times B = (C, w_C) \) by defining on the ambient set \( C = A \times B \) the weight function as \( w_C((\alpha, \beta)) = w_A(\alpha) + w_B(\beta) \). Prove that this product construction is ogf-admissible, i.e. that the ogf of the product family can be computed directly from the ogf’s of the component families. (Hint: Look at the ogf sums as defined over the structures in each of the families, so. \( c(z) = \sum_{\sigma \in C} z^{w_C(\sigma)} = \cdots \))