T-79.149 Discrete Structures, Autumn 2004

Home assignment 1 (due 20 Oct at 12:15 p.m.)

1. Solve the following recurrence equations using the technique of generating functions:

   (a)
   $$\begin{cases} a_0 = 1, \\ a_n = 2a_{n-1} + n, \quad n \geq 1; \end{cases}$$

   (b)
   $$\begin{cases} a_0 = 0, \quad a_1 = 1, \\ a_n = 4a_{n-1} - 4a_{n-2} + 2^n, \quad n \geq 2. \end{cases}$$

2. Determine the ordinary generating functions for the *harmonic sequence* $\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$ and its partial sum sequence $\langle 0, 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots \rangle$.

3. (a) Consider the formal power series $F(X) = (\mathrm{Exp}(X) - 1)/X = \sum_{n \geq 0} \frac{1}{(n+1)!} X^n$. Verify that this has an inverse $B(X) = X/(\mathrm{Exp}(X) - 1)$, and determine by formal expansion the coefficients $B_0, \ldots, B_4$ in the series $B(X) = \sum_{n \geq 0} \frac{B_n}{n!} X^n$.

   (b) The coefficients B_n determined in part (a) of the problem are called *Bernoulli numbers.* Show that they satisfy the recurrence equation
   $$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad n \geq 2.$$
   (*Hint:* Product formula for power series.)

4. Determine the ordinary generating functions for the following combinatorial families:

   (a) Nonempty unlabeled ordered rooted trees, i.e. nonempty trees that have a distinct root node, and in which the descendants of each node have a left-to-right ordering.

   (b) Binary sequences that do not contain two consequent zeroes.

   (c) Nonempty binary trees, where both subtrees of the root have the same shape.

   (d) Nonempty forests of binary trees, i.e. nonempty unordered collections of binary trees.