T-79.149 Discrete Structures (Autumn 2003)

This is an epilogue to the exercise problems for which the deadline was on October 9, 2003, i.e. to Problems 2, 10, 15 and 21 in Garzon's Section 2.6. There is nothing special to say about Problems 15 and 21.

Comment on Problem 2

It is justifiable to assume that any two groups presented by the same set of equations are isomorphic. The equations in Problem 2 do not imply commutativity of the operation in the group. They do not imply finiteness of the group either. So, it is justifiable to conclude that the group is infinite and the operation in the group is not commutative. There is still one source of uncertainty. Not knowing an "authorised" meaning for the term "modular group" in this context, one cannot exclude the possibility that modularity would form a default equation not implied by the explicit equations. However, within any fixed definition of modularity, there should not be variants of the group that would differ w.r.t. commutativity or cardinality. Considering the circumstances, answers containing such variation were tolerated with the condition that the variants were internally consistent and did not introduce weakly justifiable constraints on the operation in the group. The following paragraph is related to one of the received answers.

Not knowing the meaning of the term "associative graph", one can e.g. speculate whether it would refer to associativity of the operation in the group or to transitivity of some binary relation determined by the Cayley graph. For any group, the operation in the group is associative. So, let us consider the latter interpretation. Let S be the set of elements of a group G. For any $X \subseteq S$, let $H(X)$ denote the subgroup of G generated by X, and let $T(X)$ denote the set of elements of $H(X)$. Let $Y \subseteq S$ in such a way that $H(Y) = G$ whereas for each $z \in Y$, $z \notin T(Y \setminus \{z\})$. Let E be the set of pairs of vertices corresponding to the set of edges in the Cayley graph determined by G and Y. Let $I = \{\langle v, v\rangle \mid v \in S\}$. The following results are easy to see. (i) $E \cap I$ is empty. (ii) E and $E \cup I$ are binary relations on S. (iii) If S has more than one element, then E is not transitive. (iv) If S has more than two elements, then $E \cup I$ is not transitive.

Comment on Problem 10 (revised on November 17, 2003)

As discussed in the seminar, it was allowed to restrict the problem to Wolfram style one-dimensional automata. (The part concerning the iterates was not removed.) At that point it was not known that some kind of a restriction would have been needed anyway. We return to this subject in the end of this comment.

Formally, finite configurations are indexed as widely as infinite configurations. However, it is fair to assume an implicit correspondence between a finite word x and the bi-infinite word $(0^\omega)x(0^\omega)$. Even so, the formulation remained inconsistent in the sense that no meaningful explanation makes $T_0(C_0)$ equal to the set (hereby called $F(T(C))$) of finite subwords of image configurations in $T(C)$ in all cases. Though it is usually the responsibility of the course staff to ensure consistency of home assignments, the above inconsistency was considered to be striking enough to be included as a part of the exercise problem. Luckily, there was no terrible internal inconsistency in the answers.

Some of the answers should have had more details for making sure that the language accepted by the automaton being constructed is "close enough to exactly one of the alternatives provided by Garzon". For example, the language $T(C) \cap C_0$ is not close enough. Let $L(T(C)) = \{(0^\omega)x(0^\omega) \mid x \in F(T(C))\}$. It is fair to say that the languages $T_0(C_0)$ and $L(T(C))$ are the "alternatives provided by Garzon". It is easy to see that $T_0(C_0) \subseteq T(C) \cap C_0 \subseteq L(T(C))$. On the other hand, $(0^\omega)1(0^\omega)$ belongs to $(T(C) \cap C_0) \setminus T_0(C_0)$ e.g. in the case of the automaton 86 (cf. Problem 4 in the "deadline Oct 16" set) and to $L(T(C)) \setminus (T(C) \cap C_0)$ e.g. in the case of the automaton 254.

Wolfram's article "Computation Theory of Cellular Automata," *Communications in Mathematical Physics*, 96(1)1984, handles the alternative $L(T(C))$ in a proof-by-example style. (A web version of that proof was attached in one of the answers.) The construction in the article is primarily for $T(C)$ itself, even though the article uses the term "regular language" instead of the pedantic term "$\omega\omega$-regular language".

Regular languages are Turing-decidable. Takeo Yaku's article "The Constructibility of a Configuration in a Cellular Automaton," *Journal of Computer and System Sciences*, 7(5)1973, shows that for each integer number $d \geq 2$, there exists a d-dimensional cellular automaton such that $T_0(C_0)$ is not Turing-decidable, and that a similar results holds for $T(C) \cap C_0$.