T-79.149 Discrete Structures (Autumn 2003)

The deadline of Problems 1–4 below is on November 20, 2003, at 17:00.

Be careful with the directions of the square brackets. See Definitions 7.1 and 7.2 in [Gar] for the meaning of the terms "metric" and "continuous function between metric spaces".

Definition used by Problem 1: For any set A, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$,
$\mathcal{P}_\omega(A) = \{B \mid (B \subseteq A) \wedge (B \text{ is infinite})\}$, and $\mathcal{P}_\psi(A) = \{B \mid (B \in P_\omega(A)) \wedge ((A \setminus B) \in P_\omega(A))\}$.
We use Z to denote the set of integer numbers, and R to denote the set of real numbers. For any $a \in R$ and for any $\frown \in \{<, >, \leq, \geq\}$, we have $R_{\frown a} = \{x \mid (x \in R) \wedge (x \frown a)\}$. The usual intervals and the function ℓ from $R \times R$ to $R_{\geq 0}$ are defined by requiring that for any real numbers a and b, $[\,a, b\,] = R_{\geq a} \cap R_{\leq b}$, $]\,a, b\,] = R_{>a} \cap R_{\leq b}$, $[\,a, b\,[= R_{\geq a} \cap R_{<b}$, $]\,a, b\,[= R_{>a} \cap R_{<b}$, and $\ell(a, b) = |a - b|$. The function ζ from $\mathcal{P}(Z) \times \mathcal{P}(Z)$ to $R_{\geq 0}$ is defined by the condition
$$\forall V \in \mathcal{P}(Z) : \ (\zeta(V, V) = 0) \wedge \left(\forall W \in (\mathcal{P}(Z) \setminus \{V\}) : \zeta(V, W) = \frac{1}{1 + \min\{|k| \mid k \in (V \setminus W) \cup (W \setminus V)\}}\right).$$
Let $S \subseteq R$, and let f be a function from S to some metric space Y such that a function λ is a metric on Y. Let $x \in R$. We say that f is *left-quasicontinuous* w.r.t. λ at x if and only if
$(x \in S) \wedge (\forall \varepsilon \in R_{>0}\ \exists \delta_\varepsilon \in R_{>0}\ \forall z \in\,]\,x - \delta_\varepsilon, x\,[\ : (z \in S) \wedge (\lambda(f(z), f(x)) < \varepsilon))$.
Respectively, f is *right-quasicontinuous* w.r.t. λ at x if and only if
$(x \in S) \wedge (\forall \varepsilon \in R_{>0}\ \exists \delta_\varepsilon \in R_{>0}\ \forall z \in\,]\,x, x + \delta_\varepsilon\,[\ : (z \in S) \wedge (\lambda(f(z), f(x)) < \varepsilon))$.
Lemma used by Problem 1: ℓ is a metric on R, and ζ is a metric on $\mathcal{P}(Z)$.

A proof, though beyond the scope of homework points, is easy to make.

Problems

1. Present one pair $\langle f, g\rangle$ such that f is a bijection from $[\,0, 1\,]$ to $\mathcal{P}_\omega(Z) \cup \{\emptyset\}$ and left-quasicontinuous w.r.t. ζ at every $x \in\,]\,0, 1\,]$, g is a surjection from $\mathcal{P}(Z)$ to $[\,0, 1\,]$ and continuous in the context of ζ and ℓ, whereas the restriction of g to $\mathcal{P}_\omega(Z) \cup \{\emptyset\}$ is the inverse of f, and moreover, for each $V \in \mathcal{P}_\psi(Z) \cup \{\emptyset\}$, f is right-quasicontinuous w.r.t. ζ at $g(V)$. Show that the presented $\langle f, g\rangle$ really is as required. (6 p)

   Motivation beyond the scope of homework points: Let a function h from $\mathcal{P}(Z)$ to $\mathcal{P}(Z)$ represent the behaviour of some 2-colour 1-dimensional cellular automaton. It is easy to show that h is continuous in the context of ζ, and that consequently, $g \circ h \circ f$ is left-quasicontinuous w.r.t. ℓ at every $x \in\,]\,0,1\,]$ and right-quasicontinuous w.r.t. ℓ at least at those points where f is right-quasicontinuous w.r.t. ζ. So, the cellular automaton could in some sense be investigated by looking at some approximate curves for $g \circ h \circ f$. Sections 9.3 and 11.2 in [Gar] are closely related to this subject. Section 11.2 has some pointers to nowhere, the "8.3._" pointers being meaningfully redirectable to Section 9.3 by means of a certain recoding. See also Theorem 7.3 in [Gar] which is about the same as the famous Curtis-Hedlund-Lyndon Theorem and can be shown to hold in the context of ζ, too. In fact, ζ is a 2-colour-specific version of the metric used in the oldest known published detailed proof of the Curtis-Hedlund-Lyndon Theorem, i.e. the proof of Theorem 3.4 of [Hed] on pp. 324–325 in [Hed].

2. Let us consider Section 2 in [MazTer]. Present some one-dimensional impulse cellular automaton that constructs the basic signal illustrated by Figure 1. Explain the way of construction in this special case. (6 p)

   All needed definitions are in Section 2 in [MazTer], too.

3. Present some cellular automaton that emulates the mobile automaton on page 73 in [Wol]. Explain the mechanism of emulation in this special case. (6 p)

4. Present some cellular automaton that emulates the substitution system determined by rule (d) on page 83 in [Wol]. Explain the mechanism of emulation in this special case. (6 p)

References

[Gar] Max Garzon, *Models of Massive Parallelism: Analysis of Cellular Automata and Neural Networks*, Springer-Verlag, Berlin, Germany, 1995, xiv+272 pp.
You are supposed to be already familiar with this book.

[Hed] Gustav Arnold Hedlund, "Endomorphisms and Automorphisms of the Shift Dynamical System," *Mathematical Systems Theory*, Vol. 3, No. 4, December 1969, pp. 320–375.
The December 1969 issue can be found in the main library of HUT (in a closed stack) and in two libraries in University of Helsinki (Department of Computer Science and Department of Mathematics).

[MazTer] Jacques Mazoyer and Véronique Terrier, "Signals in One-Dimensional Cellular Automata," *Theoretical Computer Science*, Vol. 217, No. 1, March 1999, pp. 53–80.
A PDF version of the article is available e.g. via http://lib.hut.fi/.

[Wol] Stephen Wolfram, *A New Kind of Science*, Wolfram Media, Champaign, IL, USA, 2002, xiv+2+1197+67 pp.
You are supposed to be already familiar with this book.