Homework problems:

1. **Rice’s Theorem.**

   A *semantic property* of a Turing machine is any collection of languages $S$ over
   the alphabet $\{0, 1\}$. A Turing machine $M$ has the property $S$ if $L(M) \in S$.
   A property $S$ is called *trivial* if $S = \emptyset$ (no Turing machine has the property)
   or if $S = RE$ (all Turing machines have the property).

   Show that the property $S = \{L(M) \mid L(M) \text{ is a regular language}\}$ is non-trivial.
   Conclude that it is undecidable whether a Turing machine accepts a regular language.

2. Consider application programs running under some given operating system. Let
   us say that a program $P$ is *dangerous*, if it on some input modifies the operating
   system’s system files. A *general purpose virus tester* is a program that receives
   as input an arbitrary application program text $P$, analyses it and returns output
   “DANGER”, if the program is dangerous, and “OK” otherwise. Show that if any
   dangerous programs exist at all, then general-purpose virus testing is impossible.

3. Design unrestricted grammars (general rewriting systems) that generate the fol-
   lowing languages:
   
   (a) $\{w \in \{a, b, c\}^* \mid w \text{ contains equally many } a’s, b’s \text{ and } c’s\}$,
   (b) $\{a^{2^n} \mid n \geq 0\}$.

Demonstration problems:

4. Prove, without appealing to Rice’s theorem, that the following problem is unde-
   cidable:

   Given a Turing machine $M$; does $M$ accept the empty string?

5. Prove the following connections between recursive functions and languages:

   (i) A language $A \subseteq \Sigma^*$ is recursive (“Turing-decidable”), if and only its character-
       istic function

       $$
       \chi_A : \Sigma^* \to \{0, 1\}, \quad \chi_A(x) = \begin{cases} 
       1, & \text{if } x \in A; \\
       0, & \text{if } x \notin A
       \end{cases}
       $$

       is a recursive (“Turing-computable”) function.

   (ii) A language $A \subseteq \Sigma^*$ is recursively enumerable (“semidecidable”, “Turing-
        recognisable”), if and only if either $A = \emptyset$ or there exists a recursive function

       $g : \{0, 1\}^* \to \Sigma^*$ such that

       $$
       A = \{g(x) \mid x \in \{0, 1\}^*\}.
       $$