Remember to enroll for the course using the TOPI registration system by 4th of February. Registration is compulsory.

Homework problems:

1. Let $\Sigma = \{a, b\}$. Give some examples of strings from each of the following languages (at least three strings per language):
   
   (a) $\{w \in \Sigma^* \mid \text{the number of } a\text{'s in } w \text{ is even and the number of } b\text{'s is divisible of three}\}$
   
   (b) $\{a^{2n}b^m \mid n, m \geq 0\}$
   
   (c) $\{uvuRvR \mid u, v \in \Sigma^*\}$
   
   (d) $\{w \in \Sigma^* \mid \exists u, v \in \Sigma^* \text{ s.t. } w = uu = vvv\}$

2. The reversal of a string $w \in \Sigma^*$, denoted $w^R$, is defined inductively by the rules:
   
   (i) $\varepsilon^R = \varepsilon$
   
   (ii) if $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$, then $w^R = au^R$.

   It was proved in class (cf. also Lewis & Papadimitriou, p. 43) that for any strings $u, v \in \Sigma^*$ it is the case that $(uv)^R = v^Ru^R$. Prove in a similar manner, by induction based on the above definition of reversal, the following facts:

   (a) $(w^R)^R = w$
   
   (b) $(w^k)^R = (w^R)^k$, for any $k \geq 0$.

3. Prove that for any two countable sets $A_1$ and $A_2$, their union $A_1 \cup A_2$ is also countable. Deduce from this by induction that the same holds for the union of $n$ countable sets $A_1 \cup A_2 \cup \ldots \cup A_n$, for any $n \geq 2$. (Extra question: Does the claim still hold if the number of sets to be combined is countably infinite, i.e. in the case $A = A_1 \cup A_2 \cup \ldots$, where each $A_i$ is countable?)

Demonstration problems:

4. In Hilbert’s hotel there are a countably infinite number of rooms, which are numbered 0, 1, 2, 3, …. On a dark stormy night all rooms were booked, when one more traveller arrived looking for a room. How did the hotel manager solve the problem?

Later that same night, a buss with a countably infinite number of passengers parked on the hotel’s parking lot. The manager thought about the problem for a minute and managed to find rooms for all passengers. Explain how.

Having just taken care of the previous bus load of passengers, the manager was relaxing for a while, when a countably infinite number of buses, all with a countably infinite number of passengers, arrived. Could a room be found for these passengers too?

5. Show that any alphabet $\Sigma$ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over $\Sigma$ can be easily encoded into strings over $\Gamma$ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only one symbol, e.g. $\Gamma = \{1\}$?

---

1 For a definition of the notation $w^R$ see Problem 2.
6. Prove that the Cartesian product \( \mathbb{N} \times \mathbb{N} \) is countably infinite. *(Hint: Think of the pairs \((m, n) \in \mathbb{N} \times \mathbb{N}\) as embedded in the Euclidean \((x, y)\) plane \(\mathbb{R}^2\). Enumerate the pairs by diagonals parallel to the line \(y = -x\).) Conclude from this result that also the set \(\mathbb{Q}\) of rational numbers is countably infinite.

7. Let \(S\) be an arbitrary nonempty set.

(a) Give some injective (i.e. one-to-one) function \(f : S \to \mathcal{P}(S)\).

(b) Prove that there cannot exist an injective function \(g : \mathcal{P}(S) \to S\). *(Hint: Assume that such a function \(g\) existed. Consider the set \(R = \{s \in S \mid s \notin g^{-1}(s)\}\), and denote \(r = g(R)\). Is it then the case that \(r \in R\)?)*

Observe, as a consequence of item (b), that the power set \(\mathcal{P}(S)\) of any countably infinite set \(S\) is uncountable.