T-79.148 **Spring 2005**
Introduction to Theoretical Computer Science
Tutorial 2, 1–4 February
Problems

Remember to enroll for the course using the TOPI registration system by 4th of February. Registration is compulsory.

Homework problems:

1. Let $\Sigma = \{a, b\}$. Give some examples of strings from each of the following languages (at least three strings per language):

   (a) $\{w \in \Sigma^* \mid w$ the number of a's in w is even and the number of b's is divisible of three$\}$;

   (b) $\{a^{2n}b^{3m} \mid n, m \geq 0\}$;

   (c) $\{uvu^Rv^R \mid u, v \in \Sigma^*\}$;[1]

   (d) $\{w \in \Sigma^* \mid \exists u, v \in \Sigma^*$ s.t. $w = uu = vvv\}$.

2. The *reversal* of a string $w \in \Sigma^*$, denoted w^R, is defined inductively by the rules:

   (i) $\varepsilon^R = \varepsilon$;

   (ii) if $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$, then $w^R = au^R$.

   It was proved in class (cf. also Lewis & Papadimitriou, p. 43) that for any strings $u, v \in \Sigma^*$ it is the case that $(uv)^R = v^Ru^R$. Prove in a similar manner, by induction based on the above definition of reversal, the following facts:

   (a) $(w^R)^R = w$;

   (b) $(w^k)^R = (w^R)^k$, for any $k \geq 0$.

3. Prove that for any two countable sets A_1 and A_2, their union $A_1 \cup A_2$ is also countable. Deduce from this by induction that the same holds for the union of n countable sets $A_1 \cup A_2 \cup \ldots \cup A_n$, for any $n \geq 2$. (*Extra question:* Does the claim still hold if the number of sets to be combined is countably infinite, i.e. in the case $A = A_1 \cup A_2 \cup \ldots$, where each A_i is countable?)

Demonstration problems:

4. In *Hilbert's hotel* there are a countably infinite number of rooms, which are numbered 0, 1, 2, 3, ... On a dark stormy night all rooms were booked, when one more traveller arrived looking for a room. How did the hotel manager solve the problem?

   Later that same night, a buss with a countably infinite number of passengers parked on the hotel's parking lot. The manager thought about the problem for a minute and managed to find rooms for all passengers. Explain how.

   Having just taken care of the previous bus load of passengers, the manager was relaxing for a while, when a countably infinite number of buses, all with a countably infinite number of passengers, arrived. Could a room be found for these passengers too?

5. Show that any alphabet Σ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over Σ can be easily encoded into strings over Γ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only *one* symbol, e.g. $\Gamma = \{1\}$?

---
[1] For a definition of the notation w^R see Problem 2.

6. Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean (x, y) plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

7. Let S be an arbitrary nonempty set.

   (a) Give some injective (i.e. one-to-one) function $f : S \to \mathcal{P}(S)$.

   (b) Prove that there cannot exist an injective function $g : \mathcal{P}(S) \to S$. (*Hint:* Assume that such a function g existed. Consider the set $R = \{s \in S \mid s \notin g^{-1}(s)\}$, and denote $r = g(R)$. Is it then the case that $r \in R$?)

   Observe, as a consequence of item (b), that the power set $\mathcal{P}(S)$ of any countably infinite set S is uncountable.