T-79.148 **Autumn 2004**
Introduction to Theoretical Computer Science
Tutorial 9
Solutions to the demonstration problems

4. **Problem**: Prove that the class of context-free languages is not closed under intersections and complements. (*Hint:* Represent the language $\{a^k b^k c^k \mid k \geq 0\}$ as the intersection of two context-free languages.)

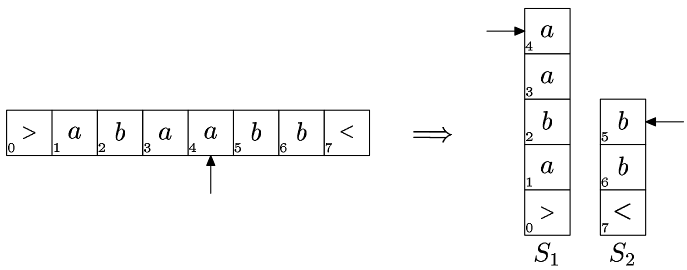
   Solution: Let $L = \{a^k b^k c^k \mid k \geq 0\}$. This language has been proven to be not context-free. We can prove that context-free languages are not closed under intersection by finding two context-free languages L_1 and L_2 such that $L = L_1 \cap L_2$. Languages $L_1 = \{a^i b^k c^k \mid i, k \geq 0\}$ and $L_2 = \{a^k b^k c^i \mid i, k \geq 0\}$ fulfill this condition.

   A direct corollary is that the class of context-free languages cannot be closed under complementation, either, since they are closed under union and $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$.

   Finally, we prove that L_1 and L_2 are context-free by presenting context-free grammars that generate them. The language L_1 is generated by $G_1 = (\{S, A, B, a, b, c\}, \{a, b, c\}, P_1, S)$, where $P_1 = \{S \to AB, A \to aA \mid \varepsilon, B \to bBc \mid \varepsilon\}$. Similarly, L_2 is generated by $G_2 = (\{S, A, B, a, b, c\}, \{a, b, c\}, P_2, S)$, $P_2 = \{S \to AB, A \to aAb \mid \varepsilon, B \to cB \mid \varepsilon\}$.

5. **Problem**: Show that pushdown automata with two stacks (rather than just one as permitted by the standard definition) would be capable of recognizing exactly the same languages as Turing machines.

   Solution: We first show that a two-stack pushdown automaton can simulate a Turing machine. The only difficulty is to find a way to simulate the Turing machine tape using two stacks. This can be done using a construction that is similar to the one presented in the first problem: the first stack holds the part of tape that is left to the read/write head (in reversed order), and the second stack holds the symbols that are right to the head.



   The computation of the automaton can be divided into two parts:

   (a) Initialization, when the automaton copies the input to stack S_1 one symbol at a time, and then moves it, again one-by-one, to stack S_2. (With the exception of the first symbol).

   (b) Simulation, where the automaton decides its next transition by examining the top symbol of S_1. If the machine moves its head to left, the top element of S_1 is moved into S_2. If it moves to the other direction, the top element of S_2 is moved to S_1.

   A two-stack pushdown automaton that is formed using these principles simulates a given Turing machine. The formal details are presented in an appendix.

   Next we show that we can simulate a two-stack pushdown automaton using a Turing machine. This can be done trivially using a two tape nondeterministic Turing machine where both stacks are stored on their own tapes.

Appendix: the formalisation of solution 5

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}}\}$ be a Turing machine. We construct a two-stack push-down automaton $M' = (Q', \Sigma', \Gamma', \delta', p_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$ as follows:

$$
\begin{aligned}
Q' =& Q \cup \{p_0, p_1, p_2\} \\
\Sigma' =& \Sigma \cup \{<\} \\
\Gamma' =& \Gamma \cup \{>, <\} \\
\delta' =& \{((p_0, \varepsilon, \varepsilon, \varepsilon), (p_1, >, \varepsilon)), ((p_1, <, \varepsilon, \varepsilon), (p_2, \varepsilon, <))\} \\
& \cup \{((p_1, x, \varepsilon, \varepsilon), (p_1, x, \varepsilon)) \mid x \in \Sigma\} \\
& \cup \{((p_2, \varepsilon, x, \varepsilon), (p_2, \varepsilon, x)) \mid x \in \Sigma\} \\
& \cup \{((q_1, \varepsilon, a, \varepsilon), (q_2, \varepsilon, b)) \mid (q_1, a, q_2, b, L) \in \delta\} \\
& \cup \{((q_1, \varepsilon, a, x), (q_2, xb, \varepsilon)) \mid (q_1, a, q_2, b, R) \in \delta, x \in \Gamma'\}
\end{aligned}
$$