T-79.148 **Autumn 2004**
Introduction to Theoretical Computer Science
Tutorial 7
Solutions to the demonstration problems

4. **Problem**: Prove that the class of context-free languages is closed under unions, concatenations, and the Kleene star operation, i.e. if the languages $L_1, L_2 \subseteq \Sigma^*$ are context-free, then so are the languages $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

   Solution: Let L_1 and L_2 be context-free languages that are defined by grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$. In addition we require that $(V_1 - \Sigma_1) \cap (V_2 - \Sigma_2) = \emptyset$. That is, the grammars may not have any common nonterminals. Since the nonterminals may be renamed if necessary, this is not an essential limitation.

   Union: Let S be a new nonterminal and $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1 \mid S_2\}, S\}$. Now $L(G) = L(G_1) \cup L(G_2) = L_1 \cup L_2$. This holds, since the initial symbol S may derive only S_1 or S_2, and they in turn may derive only strings that belong to the respective languages. (If the sets of nonterminals were not disjoint, this would not hold).

   Concatenation: The language $L_1 L_2$ is defined by the following grammar:$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1 S_2\}, S\}$

   Kleene star: The language L_1^* is defined by the following grammar: $G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \to \epsilon | S S_1\}, S\}$

5. **Problem**: Prove that the following context-free grammar is ambiguous:

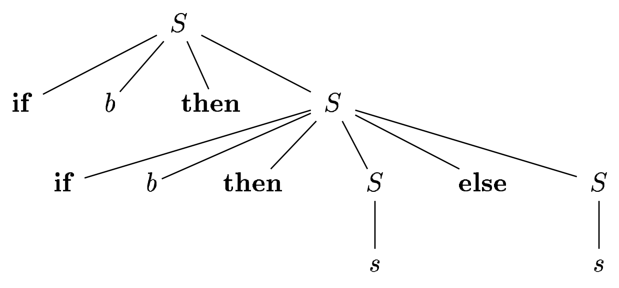
$$
\begin{aligned}
S &\to \textbf{if } b \textbf{ then } S \\
S &\to \textbf{if } b \textbf{ then } S \textbf{ else } S \\
S &\to s.
\end{aligned}
$$

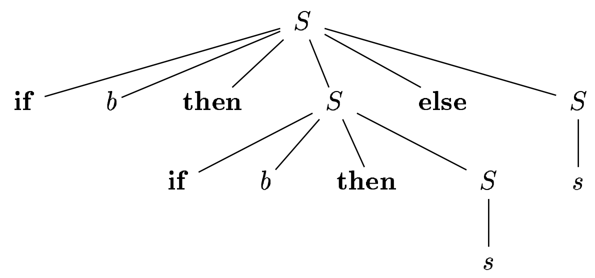
   Design an unambiguous grammar that is equivalent to the grammar, i.e. one that generates the same language.

   Solution: A context-free grammar is ambiguous if there exists a word $w \in L(G)$ such that w has at least two different parse trees. The simplest word for the given grammar that has this property is:

$$\textbf{if } b \textbf{ then if } b \textbf{ then } s \textbf{ else } s.$$

   Its two parse trees are:





Usually we want to associate an **else**-branch to the closest preceeding **if**-statement. In this case the former tree corresponds to this practice.

We define a grammar G as follows:

$$
\begin{aligned}
G &= (V, \Sigma, P, S) \\
V &= \{S, B, U, s, b, \textbf{if}, \textbf{then}, \textbf{else}\} \\
\Sigma &= \{s, b, \textbf{if}, \textbf{then}, \textbf{else}\} \\
P &= \{S \to B \mid U \\
&\quad\; B \to \textbf{if}\ b\ \textbf{then}\ B\ \textbf{else}\ B \ \mid s \\
&\quad\; U \to \textbf{if}\ b\ \textbf{then}\ S \mid \textbf{if}\ b\ \textbf{then}\ B\ \textbf{else}\ U\}
\end{aligned}
$$

Here the nonterminal B is used to derive balanced programs where each **if**-statement has both **then**- and **else**-branches. The nonterminal U derives those **if**-statements that do not have an **else**-branch.

6. **Problem**: Design a recursive-descent (top-down) parser for the grammar from Problem 6/6.

   Solution: The following C-program implements a top-down parser for the following grammar:

$$
\begin{aligned}
C &\to S \mid S; C \\
S &\to a \mid \textbf{begin}\ C\ \textbf{end} \mid \textbf{for}\ n\ \textbf{times do}\ S
\end{aligned}
$$

   This grammar is a simplified form of the one in problem 6.6. The difference is that all different numbers are replaced by a new terminal symbol n that denotes a number.

   The most important functions of the program are:

   - `C()`, `S()` — implement the rules of the program.
   - `lex()` — read the next lexeme from the input, and store it in a global variable `current_tok`.
   - `expect(int token)` — tries to read the lexeme *token* from input. Gives an error message if it fails.
   - `consume_token()` — mark the current lexeme used. This is necessary because sometimes we have to have a one-token lookahead before we know what rule we must apply.

   In practice, the programming language parsers are implemented using *lex* and *yacc* tools[1]. Of these, *lex* generates a finite automaton-based lexical analyser from identifying lexemes that have been defined using regular expression, and *yacc* constructs a pushdown automaton-based parser for a given context-free grammar.

```
#include <stdio.h>
#include <stdlib.h>
#include <ctype.h>

/* Define the alphabet */
enum TOKEN  { DO, FOR, END, BEGIN, TIMES, OP, SC, NUMBER, ERROR };
const char* tokens[] = { "do", "for", "end", "begin", "times", "a",
                         ";", "NUMBER", NULL };

/* A global variable holding the current token */
int current_tok = ERROR;
```

[1] Or some of their derivatives, like *flex* or *bison*.

```c
/* Maximum length of a token */
#define TOKEN_LEN 128

/* declare functions corresponding to nonterminals */
void S(void);
void C(void);

int lex(void);
void consume_token(void);
void error(char *st);
void expect(int token);

void C(void)
{
  S();
  lex();
  if (current_tok == SC) {
    consume_token();
    C();
    printf("C => S ; C\n");
  } else {
    printf("C => S\n");
  }
}

void S(void)
{
  lex();
  switch (current_tok) {
  case OP:
    consume_token();
    printf("S => a\n");
    break;
  case BEGIN:
    consume_token();
    C();
    expect(END);
    printf("S => begin C end\n");
    break;
  case FOR:
    consume_token();
    expect(NUMBER);
    expect(TIMES);
    expect(DO);
    S();
    printf("S => for N times do S\n");
    break;
  default:
    error("Parse error");
  }
}

/* int lex(void) returns the next token of the input. */
```

```
int lex(void)
{
  static char token_text[TOKEN_LEN];
  int pos = 0, c, i, next_token = ERROR;

  /* Is there an existing token already? */
  if (current_tok != ERROR)
    return current_tok;

  /* skip whitespace */
  do {
    c = getchar();
  } while (c != EOF && isspace(c));
  if (c != EOF) ungetc(c, stdin);

  /* read token */
  c = getchar();
  while (c != EOF && c != ';' && !isspace(c) && pos < TOKEN_LEN) {
    token_text[pos++] = c;
    c = getchar();
  }
  if (c == ';') {
    if (pos == 0) /* semicolon as token */
      next_token = SC;
    else { /* trailing semicolon, leave it for future */
      ungetc(';', stdin);
    }
  }
  token_text[pos] = '\0'; /* trailing zero */

  /* identify token */
  if (isdigit(token_text[0])) { /* number? */
    next_token = NUMBER;
  } else { /* not a number */
    for (i = DO; i < NUMBER; i++) {
      if (!strcmp(tokens[i], token_text)) {
        next_token = i;
        break;
      }
    }
  }
  current_tok = next_token;
  return next_token;
}

void consume_token(void)
{
  current_tok = ERROR;
}

void error(char *st)
{
  printf(st);
  exit(1);
```

```
}

/* try to read a 'token' from input */
void expect(int token)
{
  int next_tok = lex();
  if (next_tok == token) {
    consume_token();
    return;
  } else
    error("Parse error");
}

int main(void)
{
  int i;
  C();
  return 0;
}
```