Homework problems:

1. Let $\Sigma = \{a, b\}$. Give some examples of strings from each of the following languages (at least three strings per language):
   
   (a) $\{w \in \Sigma^* \mid \text{the number of } a\text{'s in } w \text{ is even and the number of } b\text{'s is divisible of three}\}$;
   
   (b) $\{a^{2n}b^m \mid n, m \geq 0\}$;
   
   (c) $\{uvu^Rv^R \mid u, v \in \Sigma^*\}$;
   
   (d) $\{w \in \Sigma^* \mid \exists u, v \in \Sigma^* \text{ s.t. } w = uu = vv\}$.

2. The reversal of a string $w \in \Sigma^*$, denoted $w^R$, is defined inductively by the rules:

   (i) $\varepsilon^R = \varepsilon$;
   
   (ii) if $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$, then $w^R = au^R$.

   It was proved in class (cf. also Lewis & Papadimitriou, p. 43) that for any strings $u, v \in \Sigma^*$ it is the case that $(uv)^R = v^R u^R$. Prove in a similar manner, by induction based on the above definition of reversal, the following facts:

   (a) $(w^R)^R = w$;
   
   (b) $(w^k)^R = (w^R)^k$, for any $k \geq 0$.

3. Prove that the union of two countably infinite sets is countably infinite. Deduce from this by induction that the same holds for the union of $n$ countably infinite sets, for any $n = 1, 2, \ldots$ (Extra question: Does the claim still hold if the number of sets to be combined is countably infinite, e.g. $A = A_1 \cup A_2 \cup \ldots$, where each $A_i$ is countably infinite?)

Demonstration problems:

4. Show that any alphabet $\Sigma$ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over $\Sigma$ can be easily encoded into strings over $\Gamma$ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only one symbol, e.g. $\Gamma = \{1\}$?

5. Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (Hint: Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean $(x, y)$ plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

6. Let $S$ be an arbitrary nonempty set.

   (a) Give some injective (i.e. one-to-one) function $f : S \to \mathcal{P}(S)$.
   
   (b) Prove that there cannot exist an injective function $g : \mathcal{P}(S) \to S$. (Hint: Assume that such a function $g$ existed. Consider the set $R = \{s \in S \mid s \notin g^{-1}(s)\}$, and denote $r = g(R)$. Is it then the case that $r \in R$?)

   Observe, as a consequence of item (b), that the power set $\mathcal{P}(S)$ of any countably infinite set $S$ is uncountable.

\footnote{For a definition of the notation $w^R$ see Problem 2.}