T-79.148 Autumn 2003

Introduction to Theoretical Computer Science Tutorial 9

Solutions to the demonstration problems

4. **Problem**: Prove that the class of context-free languages is not closed under intersections and complements. (*Hint:* Represent the language $\{a^kb^kc^k\mid k\geq 0\}$ as the intersection of two context-free languages.)

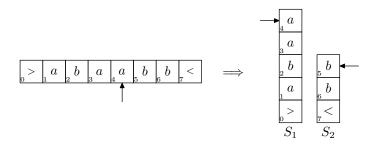
Solution: Let $L = \{a^k b^k c^k \mid k \geq 0\}$. This language has been proven to be not context-free. We can prove that context-free languages are not closed under intersection by finding two context-free languages L_1 and L_2 such that $L = L_1 \cap L_2$. Languages $L_1 = \{a^k b^k c^k \mid k \geq 0\}$ and $L_2 = \{a^k b^k c^k \mid k \geq 0\}$ fulfill this condition.

A direct corollary is that the class of context-free languages cannot be closed under complementation, either, since they are closed under union and $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$.

Finally, we prove that L_1 and L_2 are context-free by presenting context-free grammars that generate them. The language L_1 is generated by $G_1 = (\{S, A, B, a, b, c\}, \{a, b, c\}, P_1, S)$, where $P_1 = \{S \to AB, A \to aA \mid \varepsilon, B \to bBc \mid \varepsilon\}$. Similarily, L_2 is generated by $G_2 = (\{S, A, B, a, b, c\}, \{a, b, c\}, P_2, S), P_2 = \{S \to AB, A \to aAb \mid \varepsilon, B \to cB \mid \varepsilon\}$.

5. **Problem**: Show that pushdown automata with two stacks (rather than just one as permitted by the standard definition) would be capable of recognizing exactly the same languages as Turing machines.

Solution: We first show that a two-stack pushdown automaton can simulate a Turing machine. The only difficulty is to find a way to simulate the Turing machine tape using two stacks. This can be done using a construction that is similar to the one presented in the first problem: the first stack holds the part of tape that is left to the read/write head (in reversed order), and the second stack holds the symbols that are right to the head.



The computation of the automaton can be divided into two parts:

- (a) Initialization, when the automaton copies the input to stack S_1 one symbol at a time, and then moves it, again one-by-one, to stack S_2 . (With the exception of the first symbol).
- (b) Simulation, where the automaton decides its next transition by examining the top symbol of S_1 . If the machine moves its head to left, the top element of S_1 is moved into S_2 . If it moves to the other direction, the top element of S_2 is moved to S_1 .

A two-stack pushdown automaton that is formed using these principles simulates a given Turing machine. The formal details are presented in an appendix.

Next we show that we can simulate a two-stack pushdown automaton using a Turing machine. This can be done trivially using a two tape nondeterministic Turing machine where both stacks are stored on their own tapes.

Appendix: the formalisation of solution 5

Let $M=(Q,\Sigma,\Gamma,\delta,q_0,q_{\rm acc},q_{\rm rej})$ be a Turing machine. We construct a two-stack push-down automaton $M'=(Q',\Sigma',\Gamma',\delta',p_0,q_{\rm acc},q_{\rm rej})$ as follows:

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\begin{split} Q' = & Q \cup \{p_0, p_1, p_2\} \\ \Sigma' = & \Sigma \cup \{<\} \\ \Gamma' = & \Gamma \cup \{>, <\} \\ \delta' = & \{((p_0, \varepsilon, \varepsilon, \varepsilon), (p_1, >, \varepsilon)), ((p_1, <, \varepsilon, \varepsilon), (p_2, \varepsilon, <))\} \\ & \cup \{((p_1, x, \varepsilon, \varepsilon), (p_1, x, \varepsilon)) \mid x \in \Sigma\} \\ & \cup \{((p_2, \varepsilon, x, \varepsilon), (p_2, \varepsilon, x)) \mid x \in \Sigma\} \\ & \cup \{((q_1, \varepsilon, a, \varepsilon), (q_2, \varepsilon, b)) \mid (q_1, a, q_2, b, L) \in \delta\} \\ & \cup \{((q_1, \varepsilon, a, x), (q_2, xb, \varepsilon)) \mid (q_1, a, q_2, b, R) \in \delta, x \in \Gamma'\} \end{split}
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