Tik-79.148 **Spring 2001**
Introduction to Theoretical Computer Science
Tutorial 2
Answers to Demonstration Exercises

4. 1° The basic case: consider the smallest non-empty set $S_1 = \{a_1\}$. Its only partial order $R_1 = \{(a_1, a_1)\}$. (A partial order is a reflexive, anti-symmetric, and transitive binary relation.)

   An element $a \in S$ is a minimum whenever it doesn't occur as the second element of a pair in the relation (except that the reflexive self-loop is allowed). Formally, a is a minimum iff:

   $$\forall a, b \in S : (b, a) \in R \Rightarrow a = b,$$

   The element a_1 fulfills this condition in R_1 so it is a minimum.

   2° Induction hypothesis: Suppose that there exists a natural number n such that all partial orders on a set S have a minimum always when $|S| < n$.

   3° Induction step: Let $S_n = \{a_1, \ldots, a_n\}$ be a set with n elements and R_n be an arbitrary partial order on S_n. Choose now an arbitrary element $a_i \in S_n$, remove it from S_n as well as all pairs that refer to it from R_n:

   $$\begin{aligned} S'_n &= S_n - \{a_i\} \\ R'_n &= \{(a, b) \in R_n \mid a \neq a_i \wedge b \neq a_i\} \end{aligned}$$

   Now R'_n is a partial order (prove this to yourself formally, it follows from transitivity of R_n). Since $|S'_n| = n - 1 < n$, by induction hypothesis R'_n has at least one minimum that we now denote by a_{min}.

   Consider again R_n. Now there are two possibilities:

   i) If $(a_i, a_{\text{min}}) \notin R_n$, is a_{min} also a minimum of R_n.

   ii) If $(a_i, a_{\text{min}}) \in R_n$, then a_{min} can't be a minimum. However, since a_{min} is the minimum of the partial order R'_n and a partial order is always transitive, there may not be a pair $(b, a_i), b \neq a_i$ in the relation. Thus, a_i is a minimum of R_n and the induction is complete.

5. Suppose that there are n persons in the party. We try to give every one a different number of acquitances.

Person	Acquitances
1	0
2	1
3	2
$\vdots$	$\vdots$
n	$n-1$

We notice that the last person knows everybody but the first person doesn't know anybody. These two cases are in conflict, so only $n-1$ different numbers are possible. Now by the pigeon-hole principle we know that it is not possible to allocate n persons into $n-1$ slots without having at least two persons in at least one slot so it is not possible for all persons to have a different number of acquitances.

6. We can define the concatenation $v \circ w$ of strings v and w ($v, w \in \Sigma^*$) as follows:

   1° If $|v| = 0$, then $v \circ w = w$.

   2° If $|v| = n+1 > 0$, we can write v in a form $v = ua$, $u \in \Sigma^*, a \in \Sigma$. Now we define $v \circ w = u \circ aw$.

   For example, $\Sigma = \{a, b\}, v = aba, w = bba$:

$$\begin{aligned} v \circ w &= aba \circ bba \\ &= ab \circ abba \\ &= a \circ babba \\ &= e \circ ababba = ababba \end{aligned}$$

7. We have to prove that if we reverse a string twice, we get the original string. The simplest way to do it is by induction. To simplify the proof we will use the identity $(wx)^R = x^R w^R$ that is proved in the textbook.

   1° The basic case: $|w| = 0$, $(e^R)^R = e$ (by definition $e^R = e$).

   2° Induction hypothesis: Supposte that the claim holds for all $|w| \leq n, n > 0$.

   3° Induction step: Let $|w| = n+1$. Now w can be written as $w = ua$, $a \in \Sigma, u \in \Sigma^*, |u| = n$.

$$\begin{aligned} (w^R)^R &= ((ua)^R)^R \\ &= (au^R)^R \\ &= (u^R)^R (a)^R \text{ by the auxiliary identity} \\ &= (u^R)^R (ea)^R \\ &= (u^R)^R (ae^R) \\ &= (u^R)^R a \\ &= ua = w \text{ by induction hypothesis} \end{aligned}$$

8. A formal *alphabet* is a finite set of symbols. For example, the common alphabet $\{a, b, \ldots, z\}$ and the binary alphabet $\{0, 1\}$ are both also formal alphabets. Most often we use letters and numbers in alphabets, but we may also use any other symbols if necessary.

   The notation Σ^* denotes all *strings* that can be formed using the symbols in Σ including the empty string e. For example, if $\Sigma = \{a, b\}$, then $\Sigma^* = \{e, a, b, aa, ab, ba, bb, \ldots\}$. If Σ is not empty, Σ^* is necessarily infinite.

   A formal *language* L is some subset $L \subseteq \Sigma^*$. The most common notation in use is $L = \{w \in \Sigma^* \mid w \text{ fulfills the property } P\}$. That is, w is in the language if it satisfies some property P.

a) The set $L = \{w \mid \text{for some } u \in \Sigma\Sigma, w = uu^R u\}$ contains all six letter long words where the first two letters are equal to the last two letters and the middle part contains the same string reversed. The notation $u \in \Sigma\Sigma$ denotes all two-letter words.

For example, the words $abbaab$ ($u = ab$) and $aaaaaa$ ($u = aa$) belong to L. On the other hand, $w = abbbba \notin L$. Since there are only a finite number of two-letter words, L too is finite.

b) The language $L = \{w \mid ww = www\}$ contains only the empty word e. By the condition $2|w| = 3|w|$ that is only possible when $|w| = 0$ and $w = e$.

c) The language $L = \{w \mid \text{for some } u, v \in \Sigma^*, uvw = wvu\}$ contains all words ($L = \Sigma^*$). We see that if we choose $u = v = e$, then $e \circ e \circ w = w = w \circ e \circ e$ and the condition is fulfilled.

d) The language $L = \{w \mid \text{for some } u \in \Sigma^*, www = uu\}$ contains for example aa ($u = aaa$) and $aaaa$ ($u = aaaaaa$). The condition is that w is either all a or all b and $3 \cdot |w|$ has to be divisible by two. The string ab does not belong to the language.