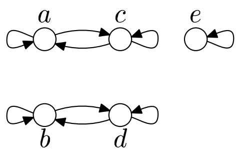
Tik-79.148 **Spring 2001**
Introduction to Theoretical Computer Science
Tutorial 1
Answers to Demonstration Exercises

4. We are given two sets, A and B, as well as a function $f : A \to B$. We then define a relation $R \subseteq A \times A$ (a relation between elements of A) with the help of B and f. A pair (a, b) is in R exactly when f maps both of them to the same element of B, that is, when $f(a) = f(b)$.

   For example, consider the case where:

$$\begin{aligned} A &= \{a, b, c, d, e\} \\ B &= \{1, 2, 3\} \\ f &= \{(a, 1), (b, 2), (c, 1), (d, 2), (e, 3)\} \ . \end{aligned}$$

   Since both $f(a) = 1$ and $f(c) = 1$, the pairs (a, c) and (c, a) are both in R. Also, $f(b) = 2 = f(d)$ so $(b, d) \in R$ and $(d, b) \in R$. Since $f(x) = f(x)$ for all elements $x \in A$, the reflexive pairs (a, a), (b, b), (c, c), (d, d), and (e, e) are all in R. The following picture shows R as a graph:



   The aim of the exercise is to show that no matter how we choose the sets A and B and the function f, the relation $R = \{(a, b) \mid f(a) = f(b)\}$ is always an equivalence relation. A relation is an equivalence when it is symmetric, transitive, and reflexive. Now we check whether the properties hold for R.

   i) A relation $R \subseteq A \times A$ is *symmetric*, if $(b, a) \in R$ always when $(a, b) \in R$. Since
   $$f(a) = f(b) \Leftrightarrow f(b) = f(a),$$
   the pair (b, a) is always in R whenever (a, b) is in it, so R is symmetric.

   ii) A relation $R \subseteq A \times A$ is *reflexive*, if for all $a \in A$ holds that $(a, a) \in R$. Because
   $$f(a) = f(a),$$
   the property holds.

   iii) A relation $R \subseteq A \times A$ is *transitive* if always when $(a, b) \in R$ and $(b, c) \in R$ it holds that $(a, c) \in R$. Intutively, a relation is transitive if two elements that are connected by some path along the arcs of the relation, are also connected by a direct arc.

   If we have:
   $$f(a) = f(b) \wedge f(b) = f(c),$$

then also

$$f(a) = f(b) = f(c) \Rightarrow f(a) = f(c),$$

so the relation is transitive.

Because all three properties hold, R is an equivalence relation.

5. A relation is a *partial order* if it is reflexive, transitive, and it doesn't have non-trivial loops (that is, $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$). We now prove that the relation $R_S = \{(A, B) \mid A, B \in S \text{ ja } A \subseteq B\}$ fulfills all three conditions:

   i) Since $A \subseteq A$, for all $A \in S : (A, A) \in R_S$ so R_S is reflexive.
   ii) Because $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$, the relation is also transitive.
   iii) The relation may have a loop only if $A \subseteq B$ and $B \subseteq A$. Then by definition $A = B$, and the loop is trivial.

6. A set A is *closed* with respect to some function[1], $f(a_1, \ldots, a_n)$ if $f(a_1, \ldots, a_n) \in A$ always when $a_1, \ldots, a_n \in A$. In other words, if the arguments of the function belong to A, the result also belongs to it. For example, the set $\mathbb{N}$ of natural numbers is closed with respect to addition but not with respect to subtraction, since $a - b$ may be negative.

   A relation B is the *closure* of R with respect to a property P if $R \subseteq B$ and B is the smallest relation that is closed with respect to P. (Two relations may be compared because they are essentially sets of ordered pairs).

   Consider the relation $R \subseteq A \times A$ where $A = \{a, b, c, d\}$ and $R = \{(a, b), (c, a)\}$. A relation is symmetric if $(b, a) \in R$ always when $(a, b) \in R$, so the property that corresponds to symmetry is the function $f_s : A \times A \to A \times A$ that reverses all pairs of the relation:

   $$f_s((x, y)) = (y, x).$$

   Now we can see that R is not symmetrically closed since, for example, $(a, b) \in R$ but $f((a, b)) = (b, a) \notin R$. We get the symmetric closure R_s of R by adding the reverse of all pairs that lack it:

   $$R_s = \{(a, b), (b, a), (c, a), (a, c)\}.$$

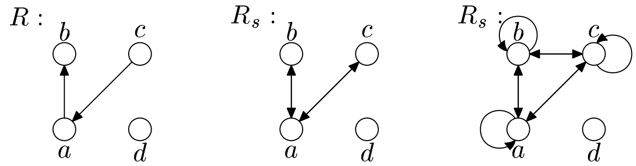
   To construct the transitive closure R_{st} we have to add the pair (x, z) whenever there are pairs $(x, y), (y, z) \in R_s$. In particular, because R_s is symmetric, we know that for all arcs (x, y) there exists a reverse arc (y, x) so $(x, x) \in R_{st}$. By adding all missing pairs we get:

   $$R_{st} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

   However, R_{st} is not reflexive, since $d \in A$, but $(d, d) \notin R_{st}$. Now we have constructed a counter example for the given claim.

   Note that R_{st} is not reflexive only in the case that A has some element a that doesn't occur in R at all. In all other cases R_{st} is reflexive.

[1] Here f is a mapping $f : B^n \to B$ where $A \subseteq B$.



7. A *partition* of a set S is a collection $P = \{P_1, \ldots, P_n\}$ of sets such that all elements of S occur in exactly one P_i and no P_i is empty. Formally: $P \subseteq 2^S$ is a partition if:

   - $P_i \neq \emptyset$ for all $1 \leq i \leq n$.
   - $P_i \cap P_j = \emptyset$ for all $i \neq j$.
   - $\bigcup P_i = S$.

   For example, the set Π of all possible partitions of $S = \{1, 2, 3\}$ is:

   $$\Pi = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\},\\ \{\{1\}\{2, 3\}\}, \{1, 2, 3\}\}\}$$

   The picture below shows how the relation R is defined among the elements of Π (the reflexive arcs are left out of the picture for clarity):

   $$\begin{tikzcd}
   & \{\{1,2\},\{3\}\} \arrow[dr] & \\
   \{\{1\},\{2\},\{3\}\} \arrow[ur] \arrow[r] \arrow[dr] \arrow[rr, bend left] & \{\{1,3\},\{2\}\} \arrow[r] & \{\{1,2,3\}\} \\
   & \{\{1\},\{2,3\}\} \arrow[ur] &
   \end{tikzcd}$$

   i) Reflexivity: Since $S_i \subseteq S_i$ for each $S_i \in \Pi_j$, the relation has the pair (Π_j, Π_j) for all Π_j $(1 \leq i \leq |\Pi_j|,\ 1 \leq j \leq |\Pi|)$.
   ii) Transitivity: If $(\Pi_i, \Pi_j) \in R$ and $(\Pi_j, \Pi_k) \in R$, then for each $S_i \in \Pi_i$ there has to exist $S_j \in \Pi_j$ such that $S_i \subseteq S_j$. From the definition of R we know that there has to be some $S_k \in \Pi_k$ such that $S_j \subseteq S_k$. Now $S_i \subseteq S_j \subseteq S_k$ so $S_i \subseteq S_k$ and there is a pair (Π_i, Π_k) in the relation.
   iii) No non-trivial loops: If $(\Pi_i, \Pi_j) \in R$ and $(\Pi_j, \Pi_i) \in R$ we know that for all $S_i \in \Pi_i$ there has to exist some $S_j \in \Pi_j$ such that $S_i \subseteq S_j$. On the other hand, there also has to be some $S_i' \in \Pi_i$ where $S_j \subseteq S_i'$. From this we get that $S_i \subseteq S_i'$. Since by definition all sets in a partition Π_i are nonempty and all elements of S are in exactly one set S_i, the only possibility is that $S_i = S_i'$ and:

   $$S_i \subseteq S_j \subseteq S_i.$$

   This implies that $S_i = S_j$ and that $\Pi_i = \Pi_j$ so the loop is trivial.

   The maximum element of R is the trivial partition that has S as its only element. The minimum element is a partition where all elements of S belong to different partitions.