4. We are given two sets, $A$ and $B$, as well as a function $f : A \rightarrow B$. We then define a relation $R \subseteq A \times A$ (a relation between elements of $A$) with the help of $B$ and $f$. A pair $(a, b)$ is in $R$ exactly when $f$ maps both of them to the same element of $B$, that is, when $f(a) = f(b)$.

For example, consider the case where:

\[
\begin{align*}
A &= \{a, b, c, d, e\} \\
B &= \{1, 2, 3\} \\
f &= \{(a, 1), (b, 2), (c, 1), (d, 2), (e, 3)\}
\end{align*}
\]

Since both $f(a) = 1$ and $f(c) = 1$, the pairs $(a, c)$ and $(c, a)$ are both in $R$. Also, $f(b) = 2 = f(d)$ so $(b, d) \in R$ and $(d, b) \in R$. Since $f(x) = f(x)$ for all elements $x \in A$, the reflexive pairs $(a, a)$, $(b, b)$, $(c, c)$, $(d, d)$, and $(e, e)$ are all in $R$. The following picture shows $R$ as a graph:

The aim of the exercise is to show that no matter how we choose the sets $A$ and $B$ and the function $f$, the relation $R = \{(a, b) \mid f(a) = f(b)\}$ is always an equivalence relation. A relation is an equivalence when it is symmetric, transitive, and reflexive. Now we check whether the properties hold for $R$.

i) A relation $R \subseteq A \times A$ is **symmetric**, if $(b, a) \in R$ always when $(a, b) \in R$. Since $f(a) = f(b) \iff f(b) = f(a)$, the pair $(b, a)$ is always in $R$ whenever $(a, b)$ is in it, so $R$ is symmetric.

ii) A relation $R \subseteq A \times A$ is **reflexive**, if for all $a \in A$ holds that $(a, a) \in R$. Because $f(a) = f(a)$, the property holds.

iii) A relation $R \subseteq A \times A$ is **transitive** if always when $(a, b) \in R$ and $(b, c) \in R$ it holds that $(a, c) \in R$. Intuitively, a relation is transitive if two elements that are connected by some path along the arcs of the relation, are also connected by a direct arc. If we have:

$$f(a) = f(b) \land f(b) = f(c),$$

1
then also

\[ f(a) = f(b) = f(c) \Rightarrow f(a) = f(c), \]

so the relation is transitive.

Because all three properties hold, \( R \) is an equivalence relation.

5. A relation is a partial order if it is reflexive, transitive, and it doesn’t have non-trivial loops (that is, \((a, b) \in R \) and \((b, a) \in R \) implies that \( a = b \)). We now prove that the relation \( R_S = \{(A, B) \mid A, B \in S \text{ for all } A \subseteq B \} \) fulfills all three conditions:

   i) Since \( A \subseteq A \), for all \( A \in S : (A, A) \in R_S \) so \( R_S \) is reflexive.
   
   ii) Because \( A \subseteq B, B \subseteq C \Rightarrow A \subseteq C \), the relation is also transitive.
   
   iii) The relation may have a loop only if \( A \subseteq B \) and \( B \subseteq A \). Then by definition \( A = B \), and the loop is trivial.

6. A set \( A \) is closed with respect to some function\(^1\), \( f(a_1, \ldots, a_n) \) if \( f(a_1, \ldots, a_n) \in A \) always when \( a_1, \ldots, a_n \in A \). In other words, if the arguments of the function belong to \( A \), the result also belongs to it. For example, the set \( \mathbb{N} \) of natural numbers is closed with respect to addition but not with respect to subtraction, since \( a - b \) may be negative.

A relation \( B \) is the closure of \( R \) with respect to a property \( P \) if \( R \subseteq B \) and \( B \) is the smallest relation that is closed with respect to \( P \). (Two relations may be compared because they are essentially sets of ordered pairs).

Consider the relation \( R \subseteq A \times A \) where \( A = \{a, b, c, d\} \) and \( R = \{(a, b), (c, a)\} \). A relation is symmetric if \((b, a) \in R \) always when \((a, b) \in R \), so the property that corresponds to symmetry is the function \( f_s : A \times A \rightarrow A \times A \) that reverses all pairs of the relation:

\[ f_s((x, y)) = (y, x). \]

Now we can see that \( R \) is not symmetrically closed since, for example, \((a, b) \in R \) but \( f((a, b)) = (b, a) \notin R \). We get the symmetric closure \( R_s \) of \( R \) by adding the reverse of all pairs that lack it:

\[ R_s = \{(a, b), (b, a), (c, a), (a, c)\}. \]

To construct the transitive closure \( R_{st} \) we have to add the pair \((x, z)\) whenever there are pairs \((x, y), (y, z) \in R_s \). In particular, because \( R_s \) is symmetric, we know that for all arcs \((x, y)\) there exists a reverse arc \((y, x)\) so \((x, x) \in R_{st} \). By adding all missing pairs we get:

\[ R_{st} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}. \]

However, \( R_{st} \) is not reflexive, since \( d \in A \), but \((d, d) \notin R_{st} \). Now we have constructed a counter example for the given claim.

Note that \( R_{st} \) is not reflexive only in the case that \( A \) has some element \( a \) that doesn’t occur in \( R \) at all. In all other cases \( R_{st} \) is reflexive.

\(^1\)Here \( f \) is a mapping \( f : B^n \rightarrow B \) where \( A \subseteq B \).
7. A partition of a set $S$ is a collection $P = \{P_1, \ldots, P_n\}$ of sets such that all elements of $S$ occur in exactly one $P_i$ and no $P_i$ is empty. Formally: $P \subseteq 2^S$ is a partition if:

- $P_i \neq \emptyset$ for all $1 \leq i \leq n$.
- $P_i \cap P_j = \emptyset$ for all $i \neq j$.
- $\bigcup P_i = S$.

For example, the set $\Pi$ of all possible partitions of $S = \{1, 2, 3\}$ is:

$$\Pi = \{\{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1, 2\}\}, \{\{1, 3\}\}, \{\{2, 3\}\}, \{\{1, 2, 3\}\}\}$$

The picture below shows how the relation $R$ is defined among the elements of $\Pi$ (the reflexive arcs are left out of the picture for clarity):

![Diagram showing the relation R among partitions]

i) Reflexivity: Since $S_i \subseteq S_i$ for each $S_i \in \Pi_j$, the relation has the pair $(\Pi_j, \Pi_j)$ for all $\Pi_j$ (1 ≤ $i$ ≤ |$\Pi_j$|, 1 ≤ $j$ ≤ |$\Pi$|).

ii) Transitivity: If $(\Pi_i, \Pi_j) \in R$ and $(\Pi_j, \Pi_k) \in R$, then for each $S_i \in \Pi_i$ there has to exist $S_j \in \Pi_j$ such that $S_i \subseteq S_j$. From the definition of $R$ we know that there has to be some $S_k \in \Pi_k$ such that $S_j \subseteq S_k$. Now $S_i \subseteq S_j \subseteq S_k$ so $S_i \subseteq S_k$ and there is a pair $(\Pi_i, \Pi_k)$ in the relation.

iii) No non-trivial loops: If $(\Pi_i, \Pi_j) \in R$ and $(\Pi_j, \Pi_i) \in R$ we know that for all $S_i \in \Pi_i$ there has to exist some $S_j \in \Pi_j$ such that $S_i \subseteq S_j$. On the other hand, there also has to be some $S'_j \in \Pi_i$ where $S_j \subseteq S'_j$. From this we get that $S_i \subseteq S'_j$. Since by definition all sets in a partition $\Pi_i$ are nonempty and all elements of $S$ are in exactly one set $S_i$, the only possibility is that $S_i = S'_j$ and:

$$S_i \subseteq S_j \subseteq S_i.$$  

This implies that $S_i = S_j$ and that $\Pi_i = \Pi_j$ so the loop is trivial.

The maximum element of $R$ is the trivial partition that has $S$ as its only element. The minimum element is a partition where all elements of $S$ belong to different partitions.