**Tik-79.146** **Kevät 1999**
**Logiikka tietotekniikassa: erityiskysymyksiä I**
**Laskuharjoitus 6**
**Ratkaisut**

Aksioomat ja päättelysäännöt:

$$\begin{aligned}
\text{K:}\quad & \Box(P \to Q) \to (\Box P \to \Box Q) \\
\text{T:}\quad & \Box P \to P \\
\text{D:}\quad & \Box P \to \Diamond P \quad \text{tai} \quad \Box P \to \neg\Box\neg P \\
\text{4:}\quad & \Box P \to \Box\Box P \\
\text{5:}\quad & \neg\Box P \to \Box\neg\Box P \\
\text{MP:}\quad & \frac{P,\ P \to Q}{Q} \\
\text{N:}\quad & \frac{P}{\Box P}
\end{aligned}$$

1. a)

| | | |
|---|---|---|
| 1. | $P \to (Q \to P)$ | [Tautologia] |
| 2. | $\Box\big(P \to (Q \to P)\big)$ | [N, 1] |
| 3. | $\Box\big(P \to (Q \to P)\big) \to \big(\Box P \to \Box(Q \to P)\big)$ | [K] |
| 4. | $\Box P \to \Box(Q \to P)$ | [MP, 2,3] |

b)

| | | |
|---|---|---|
| 1. | $\Box(P \to Q)$ | [GP] |
| 2. | $(P \to Q) \to (\neg Q \to \neg P)$ | [Taut] |
| 3. | $\Box\big((P \to Q) \to (\neg Q \to \neg P)\big)$ | [N, 2] |
| 4. | $\Box\big((P \to Q) \to (\neg Q \to \neg P)\big) \to \big(\Box(P \to Q) \to \Box(\neg Q \to \neg P)\big)$ | [K] |
| 5. | $\Box(P \to Q) \to \Box(\neg Q \to \neg P)$ | [MP, 3,4] |
| 6. | $\Box(\neg Q \to \neg P)$ | [MP, 1,5] |
| 7. | $\Box(\neg Q \to \neg P) \to (\Box\neg Q \to \Box\neg P)$ | [K] |
| 8. | $\Box\neg Q \to \Box\neg P$ | [MP, 6,7] |
| 9. | $(\Box\neg Q \to \Box\neg P) \to (\neg\Box\neg P \to \neg\Box\neg Q)$ | [Taut] |
| 10. | $\neg\Box\neg P \to \neg\Box\neg Q$ | [MP, 8,9] |

2. a)

| | | |
|---|---|---|
| 1. | $P \to Q$ | [GP] |
| 2. | $\neg Q \to P$ | [GP] |
| 3. | $(P \to Q) \to \big((\neg Q \to P) \to Q\big)$ | [Taut] |
| 4. | $(\neg Q \to P) \to Q$ | [MP, 1,3] |
| 5. | $Q$ | [MP, 2,4] |
| 6. | $\Box Q$ | [N, 5] |
| 7. | $\neg Q \vee S$ | [LP] |
| 8. | $(\neg Q \vee S) \to (Q \to S)$ | [Taut] |
| 9. | $Q \to S$ | [MP, 7,8] |
| 10. | $S$ | [MP, 5,9] |
| 11. | $\Box Q \to (S \to \Box Q \wedge S)$ | [Taut] |
| 12. | $S \to \Box Q \wedge S$ | [MP, 6,11] |
| 13. | $\Box Q \wedge S$ | [MP, 10,12] |

b)

| | | |
|---|---|---|
| 1. | $Q \to \neg P$ | [GP] |
| 2. | $\Box(Q \to \neg P)$ | [N, 1] |
| 3. | $\Box(Q \to \neg P) \to (\Box Q \to \Box\neg P)$ | [K] |
| 4. | $\Box Q \to \Box\neg P$ | [MP, 2, 3] |
| 5. | $\Diamond Q \to \Box Q$ | [GP] |
| 6. | $(\Diamond Q \to \Box Q) \to$ $\big((\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)\big)$ | [Taut] |
| 7. | $(\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)$ | [MP, 5, 6] |
| 8. | $\Diamond Q \to \Box\neg P$ | [MP, 4, 7] |
| 9. | $(\neg\Box\neg Q \to \Box\neg P) \to (\neg\Box\neg P \to \Box\neg Q)$ | [Taut] |
| 10. | $\neg\Box\neg P \to \Box\neg Q$ | [MP, 8, 9] |
| 11. | $\Diamond P$ | [LP] |
| 12. | $\Box\neg Q$ | [MP, 10, 11] |

3.

| | | |
|---|---|---|
| 1. | $\neg\Box P \to \Box\neg\Box P$ | [5] |
| 2. | $(\neg\Box P \to \Box\neg\Box P) \to (\neg\Box\neg\Box P \to \Box P)$ | [Taut] |
| 3. | $\neg\Box\neg\Box P \to \Box P$ | [MP, 1,2] |
| 4. | $\Box\Box P$ | [GP] |
| 5. | $\Box\Box P \to \neg\Box\neg\Box P$ | [D] |
| 6. | $\neg\Box\neg\Box P$ | [MP, 4,5] |
| 7. | $\Box P$ | [MP, 6,3] |

4. a) Huomataan että

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge A \wedge B \wedge \neg(A \wedge B)\big),$$
$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge \neg A \wedge (A \wedge B)\big)$$

ja

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge \neg B \wedge (A \wedge B)\big)$$

koska ne ovat tautologioita. Koska $\Delta$ on $\Sigma$-maksimaalinen, tämä implikoi että $A \wedge B \notin \Delta$, $A \in \Delta$ ja $B \in \Delta$ ei ole mahdollista, eikä $A \wedge B \in \Delta$ ja $A \notin \Delta$, eikä $A \wedge B \in \Delta$ ja $B \notin \Delta$. Siis, $A \wedge B \in \Delta$ joss $A \in \Delta$ ja $B \in \Delta$.

b) Samasta syystä

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge \neg A \wedge \neg(A \to B)\big),$$
$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge B \wedge \neg(A \to B)\big),$$

ja

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge A \wedge \neg B \wedge (A \to B)\big)$$

eli $A \to B \in \Delta$ joss $A \notin \Delta$ tai $B \in \Delta$.

c) Vastaavasti

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge A \wedge \neg(A \vee B)\big),$$
$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge B \wedge \neg(A \vee B)\big),$$

ja

$$\Sigma \vdash \emptyset \Longrightarrow \neg\big(\top \wedge \neg A \wedge \neg B \wedge (A \vee B)\big)$$

ja $A \vee B \in \Delta$ joss $A \in \Delta$ tai $B \in \Delta$.

5. **S4** on **KT4**.

| | | |
|---|---|---|
| 1. | $\neg\bot$ | [Taut] |
| 2. | $\Box\bot \to \bot$ | [T] |
| 3. | $(\Box\bot \to \bot) \to (\neg\bot \to \neg\Box\bot)$ | [Taut] |
| 4. | $\neg\bot \to \neg\Box\bot$ | [MP, 2, 3] |
| 5. | $\neg\Box\bot$ | [MP, 1, 4] |