Tik-79.144 **Syksy 2000**
Logic in Computer Science: Foundations
Exercises 4
Solutions

1. a) First equivalences and implications are removed from the proposition, and then the negations are transferred to the "innermost" level using De Morgan rules.

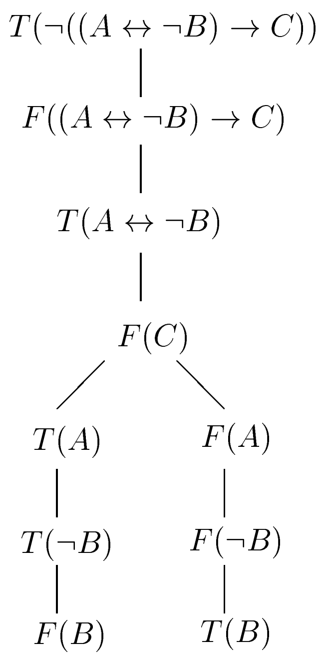
$$
\begin{aligned}
&\neg((A \leftrightarrow \neg B) \to C) \\
&\qquad \equiv \neg((A \to \neg B) \wedge (\neg B \to A) \to C) && [\leftrightarrow e] \\
&\qquad \equiv \neg(\neg((\neg A \vee \neg B) \wedge (\neg\neg B \vee A)) \vee C) && [\to e] \\
&\qquad \equiv (\neg A \vee \neg B) \wedge (A \vee B) \wedge \neg C \; (*) && [\neg\neg e, DM]
\end{aligned}
$$

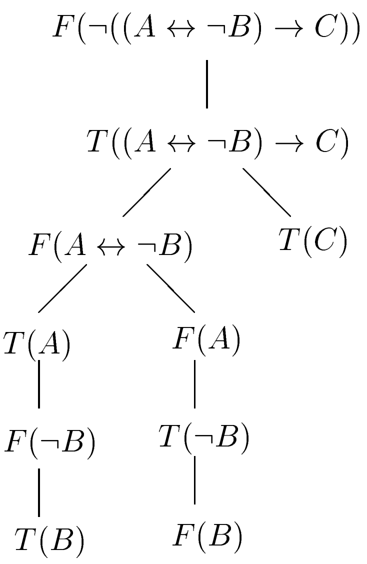
Now the conjunctive normal form is found. The disjunctive normal form is found by using the distributivity of conjunction over distribution (ie. $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$).

$$
\begin{aligned}
(*) &\equiv (\neg A \vee \neg B) \wedge ((B \wedge \neg C) \vee (A \wedge \neg C)) \\
&\equiv (\neg A \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) \vee \\
&\qquad (\neg B \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) \\
&\equiv (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge A \wedge \neg C) \vee \\
&\qquad (\neg B \wedge B \wedge \neg C) \vee (\neg B \wedge A \wedge \neg C) \\
&\equiv (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)
\end{aligned}
$$

Now the disjunctive normal form is found. In the last step unsatisfiable conjunctions were left out.

b) It is easy to find all the models of a proposition by using its semantic tableau, and the disjunctive normal form can be constructed from these models. We'll construct semantic tableaux for both the proposition and its negation:





The disjunctive normal form can be read from the *open* brances of the first tableau: $(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$. There is one conjunction for each open branch. If there is a node $T(A)$ in the branch, then the proposition A will be in the conjunction, and if there is a node $F(A)$, proposition $\neg A$ is used.

From the other tableau a disjunctive normal form can be found for the negation of the original proposition: $(A \wedge B) \vee (\neg A \wedge \neg B) \vee C$. By taking the negation $\neg((A \wedge B) \vee (\neg A \wedge \neg B) \vee C)$ we have a proposition that is equivalent with the original proposition. Now we can use the De Morgan rules to construct the conjunctive normal form for the original proposition: $(\neg A \vee \neg B) \wedge (A \vee B) \wedge \neg C$.

You should check, that the results agrees with results from (a).

2. The notation $\models \phi$ means that ϕ is valid. So in both cases the claim is that the propositions are valid. We use semantic tableaux to examine the validity of the propositions in normal way:

a) $F((A \to (B \to C)) \to ((A \to C) \to (A \to B)))^1$

$$
\begin{array}{c}
F((A \to (B \to C)) \to ((A \to C) \to (A \to B)))^1 \\
| \\
T(A \to (B \to C))^5 \\
| \\
F((A \to C) \to (A \to B))^2 \\
| \\
T(A \to C)^4 \\
| \\
F(A \to B)^3 \\
| \\
T(A) \\
| \\
F(B) \\
\begin{array}{cc}
\begin{array}{c} F(A) \\ \otimes \end{array} &
\begin{array}{c} T(C) \\ \begin{array}{cc} F(A) & \begin{array}{c} T(B \to C)^6 \\ \begin{array}{cc} F(B) & T(C) \end{array} \end{array} \end{array} \end{array}
\end{array}
\end{array}
$$

Because the tableau is not contradictionary, the proposition is not valid. The counter example can be read from an open branch of the tableau. If for some atomic symbol P there is $T(P)$ in the branch we add $\mathcal{V}(P) = T$ to the counter example, and if there is $F(P)$, we add $\mathcal{V}(P) = F$. From the rightmost branch we get:

$$\mathcal{V}(A) = \mathcal{V}(C) = T, \mathcal{V}(B) = F$$

b) $F((\neg B \to (A \to C)) \to (A \to B \vee C))^1$

$$
\begin{array}{c}
F((\neg B \to (A \to C)) \to (A \to B \vee C))^1 \\
| \\
T(\neg B \to (A \to C))^4 \\
| \\
F(A \to B \vee C)^2 \\
| \\
T(A) \\
| \\
F(B \vee C)^3 \\
| \\
F(B) \\
| \\
F(C) \\
\begin{array}{cc}
\begin{array}{c} F(\neg B)^5 \\ | \\ T(B) \\ \otimes \end{array} &
\begin{array}{c} T(A \to C)^6 \\ \begin{array}{cc} \begin{array}{c} F(A) \\ \otimes \end{array} & \begin{array}{c} T(C) \\ \otimes \end{array} \end{array} \end{array}
\end{array}
\end{array}
$$

As all branches are contradictionary the proposition is valid, and the claim is true.

3. As C is a resolvent of C_1 and C_2, there are l, C_1', and C_2' such that $C_1 = \{l\} \cup C_1'$, $C_2 = \{\bar{l}\} \cup C_2'$, and $C = C_1' \cup C_2'$.

Let $\mathcal{A}$ be a truth assignment that satisfies both C_1 and C_2. Now both clauses have at least one literal in them that is true in the truth assignment. By definition of a truth assignment both l and $\bar{l}$ may not be true in the assignment.

We examine first the case where $l \in \mathcal{A}$. Because $\mathcal{A} \models C_2$ and $\bar{l} \notin A$, $\mathcal{A} \models C_2'$ ($\mathcal{A}$ (a literal other than $\bar{l}$ is true in the assignment). Because C_2' is a subset of C, $\mathcal{A} \models C$.

For the case of $\bar{l} \in \mathcal{A}$ the proof can be constructed as above by replacing C_2 by C_1.