Tik-79.144 **Syksy 2000**
Logic in Computer Science: Foundations
Exercise 10
Solutions

1.

$$T(\exists x(P(x) \to Q(x)))^{1\{x/c\}}$$
$$T(\exists x(\neg R(x) \to \neg Q(x)))^{2\{x/d\}}$$
$$F(\exists x(P(x) \to R(x)))^{3\{x/c\},6\{x/d\}}$$
$$T(P(c) \to Q(c))^{5}$$
$$T(\neg R(d) \to \neg Q(d))^{8}$$
$$F(P(c) \to R(c))^{4}$$
$$T(P(c))$$
$$F(R(c))$$
$$\begin{array}{cc}
F(P(c)) & T(Q(c)) \\
\otimes & F(P(d) \to R(d))^{7} \\
 & T(P(d)) \\
 & F(R(d))
\end{array}$$
$$\begin{array}{cc}
F(\neg R(d)) & T(\neg Q(d)) \\
T(R(d)) & F(Q(d)) \\
\otimes &
\end{array}$$

From the open branch we can read the counter-example:

$$\mathcal{A} = \{c, d\}$$
$$P^{\mathcal{A}} = \{c, d\}$$
$$R^{\mathcal{A}} = \emptyset$$
$$Q^{\mathcal{A}} = \{c\}$$

2. a) Define predicates $R_n(x,y)$ as follows:

$$\begin{gathered}\forall x R_0(x,x)\\ \forall x\forall y\forall z(R_0(x,y)\wedge K(y,z)\to R_1(x,z))\\ \forall x\forall y\forall z(R_1(x,y)\wedge K(y,z)\to R_2(x,z))\\ \vdots\\ \forall x\forall y\forall z(R_{k-1}(x,y)\wedge K(y,z)\to R_k(x,z))\end{gathered}$$

The rules mean that each node has a route of length 0 to itself, and if there is a route of length $k-1$ from node x to node y and there is an edge from y to z, then there is a route of length k from x to z.

Graph:

$$a \;\underset{\longleftarrow}{\overset{\longrightarrow}{}}\; b \longrightarrow c$$

can be formalized as follows:

$$\begin{gathered}K(a,b)\\ K(b,a)\\ K(b,c)\end{gathered}$$

b) To use resolution we have to transform the set of sentences into clausul form.

From the definition of the graph we get clauses

$$\{K(a,b)\},\{K(b,a)\},\{K(b,c)\}$$

Because we want to examine routes of length 3 we have to define predicates R_n only up to value $n=3$.

$$\begin{gathered}\{R_0(x,x)\}\\ \{\neg R_0(x,y),\neg K(y,z),R_1(x,z)\}\\ \{\neg R_1(x,y),\neg K(y,z),R_2(x,z)\}\\ \{\neg R_2(x,y),\neg K(y,z),R_3(x,z)\}\end{gathered}$$

Finally add the negation of the desired consequence to the graph

$$\begin{aligned}&\neg\exists x(R_2(x,x)\wedge R_3(x,c))\\ \equiv\;&\forall x\neg(R_2(x,x)\wedge R_3(x,c))\\ \equiv\;&\forall x(\neg R_2(x,x)\vee\neg R_3(x,c))\\ \Rightarrow\;&\{\neg R_2(x,x),\neg R_3(x,c))\}\end{aligned}$$

Informally, the desired consequence is "there is some node such that there is a loop of length 2 back to the node and a path of length 3 to the node c". When we look at the graph we see that node b fulfills this condition. The easiest way to construct the proof is by doing it in following steps:

1. Prove that there is a route of length 1 from b to a.
2. Prove that there is a route of length 2 from b to b.
3. Prove that there is a route of length 3 from b to c.

1	$\{K(b,a)\}$	P
2	$\{\neg R_0(x_1,y_1), \neg K(y_1,z_1), R_1(x_1,z_1)\}$	P
3	$\{\neg R_0(x_1,b), R_1(x_1,a)\}$	$[1,2]$ MGU: $\{y_1/b, z_1/a\}$
4	$\{R_0(x_2,x_2)\}$	P
5	$\{R_1(b,a)\}$	$[3,4]$ MGU: $\{x_1/b, x_2/b\}$
6	$\{\neg R_1(x_3,y_3), \neg K(y_3,z_3), R_2(x_3,z_3)\}$	P
7	$\{\neg K(a,z_3), R_2(b,z_3)\}$	$[5,6]$ MGU: $\{x_3/b, y_3/a\}$
8	$\{K(a,b)\}$	P
9	$\{R_2(b,b)\}$	$[7,8]$ MGU: $\{z_3/b\}$
10	$\{\neg R_2(x,y), \neg K(y,z), R_3(x,z)\}$	P
11	$\{\neg K(b,z_4), R_3(b,z_4)\}$	$[9,10]$ MGU: $\{x_4/b, y_4/b\}$
12	$\{K(b,c)\}$	P
13	$\{R_3(b,c)\}$	$[11,12]$ MGU: $\{z_4/c\}$
14	$\{\neg R_2(x_5,x_5), \neg R_3(x_5,c)\}$	P
15	$\{\neg R_3(b,c)\}$	$[9,14]$ MGU: $\{x_5/b\}$
16	$\square$	$[13,15]$