Tik-79.144 **Syksy 1999**
Logic in Computer Science: Foundations
Exercises 6
Solutions

1. All of the solutions use either universal or existential quantifiers or both. If we want to say that some property $\phi(x)$ holds for all those x that have also property $P(x)$, we formalize that with: $\forall x(P(x) \rightarrow \phi(x))$. If some property $\phi(x)$ holds for some x that also satisfies $P(x)$, we use $\exists x(P(x) \wedge \phi(x))$. In the solutios many predicates (e.g. $P(x)$ in the first case) are used to denote the type of the x.

a) $\exists x(P(x) \wedge V(x))$, when
$P(x) = x$ is a gate.
$V(x) = x$ is faulty.

b) $A(a) \wedge (\forall x(A(x) \wedge \neg(x = a) \rightarrow N(a, x))$, when
$a =$ the algorithm in question
$A(x) = x$ is an algorithm
$N(x, y) = x$ is faster than y.

c) $\forall x(K(x) \rightarrow \exists y(T(y) \wedge R(x, y)))$, when
$K(x) = x$ is a participant of the course.
$T(x) = x$ is a workstation
$R(x, y) = x$ uses y.

d) $\forall x(T(x) \rightarrow \forall y \forall z(P(y) \wedge P(z) \wedge K(y, x) \wedge K(z, x) \rightarrow y = z))$, when
$P(x) = x$ is a process.
$T(x) = x$ is a file.
$K(x, y) = x$ writes in y.

The above solutions are not the only possible ones.

2. a) In the graphs we are particularly interested in edges, which we will denote by predicate $K(x, y)$ (there is a edge from node x to node y in the graph). There are several possible ways to denote the colors.

(i) We can fix the set of the colors and represent them as predicates. If there are n different colors in set C, we define predicates $C_1(x), \ldots, C_n(n)$. A predicate $C_i(x)$ means that the node x is of the color C_i. The problem description demands that each node has a unique color and that if there is a edge between two nodes the nodes have different colors.

The first condition can be stated with a set of statements of the form:

$$\forall x(C_i(x) \leftrightarrow \neg C_1(x) \wedge \cdots \wedge \neg C_{i-1}(x) \wedge \neg C_{i+1} \wedge \cdots \wedge \neg C_n(x))$$

where $i = 1, \ldots, n$ (notice that $\neg C_i(x)$ is not in the conjunction of the right side).

The second condition is formalized for each $C_i(x)$ as follows:

$$\forall x \forall y(K(x, y) \rightarrow (C_i(x) \rightarrow \neg C_i(y))).$$

(ii) The second possibility is to leave the definition of the colors open and use a predicate $V(x, y)$ (the node x is of the color y).

Now the uniqueness of node colors can be expressed as:

$$\forall x \forall y \forall z(V(x, y) \wedge V(x, z) \rightarrow y = z).$$

Informally, if a node x has both colors y and x, then the colors y and z must, in fact, be the same color.

The second condition can be expressed with:

$$\forall x \forall y \forall z(K(x, y) \rightarrow (V(x, z) \rightarrow \neg V(y, z)).$$

(iii) The third possibility is to define a function symbol v. Now $v(x)$ means the color of the node x. Because the value of a function is by definition unique, only the second condition has to be formalized:

$$\forall x \forall y(K(x, y) \rightarrow \neg(v(x) = v(y))).$$

b) Let's construct a model for the case (i) when $n = 2$. We will define a structure $\mathcal{A}$, where the universume is $A = \{a_1, a_2\}$ (two nodes). The interpretation of predicate K is $K^{\mathcal{A}} = \{\langle a_1, a_2 \rangle, \langle a_2, a_1 \rangle\}$ (there is a edge from node a_1 to a_2 and from a_2 to a_1).

The interpretation of the colors C_1 and C_2 are $C_1^{\mathcal{A}} = \{a_1\}$ and $C_2^{\mathcal{A}} = \{a_2\}$. We must also define two constant symbols s_1 and s_2, which denote the atoms a_1 and a_2. They are needed because the definition of a predicate logic structure demands that there is a variable-free term denoting each atom of the universume A.

We now check that sentences

$$\forall x(C_1(x) \leftrightarrow \neg C_2(x))$$

$$\forall x \forall y (K(x,y) \to (C_1(x) \to \neg C_1(y))$$

and

$$\forall x \forall y (K(x,y) \to (C_2(x) \to \neg C_2(y))$$

are true in the structure. $\mathcal{A}$ (that is, $\mathcal{A}$ is a model for the sentences). The first of the sentences is equivalent to

$$\forall x (C_2(x) \leftrightarrow \neg C_1(x)),$$

which also belongs to the set of sentences when $n = 2$.
Because s_1 and s_2 are only ground terms,

$$\mathcal{A} \models \forall x (C_1(x) \leftrightarrow \neg C_2(x))$$

if and only if

$$\mathcal{A} \models (C_1(s_1) \leftrightarrow \neg C_2(s_1)) \qquad \text{and} \qquad \mathcal{A} \models (C_1(s_2) \leftrightarrow \neg C_2(s_2)).$$

Because $s_1^{\mathcal{A}} \in C_1^{\mathcal{A}}$,

$$\mathcal{A} \models C_1(s_1)$$

and because $s_1^{\mathcal{A}} \notin C_2^{\mathcal{A}}$,

$$\mathcal{A} \not\models C_2(s_1)$$

So

$$\mathcal{A} \models (C_1(s_1) \leftrightarrow \neg C_2(s_1)).$$

We can show respectively that $\mathcal{A} \models (C_1(s_2) \leftrightarrow \neg C_2(s_2))$, and so $\mathcal{A} \models \forall x (C_1(x) \leftrightarrow \neg C_2(x))$ follows.
Respectively,

$$\mathcal{A} \models \forall x \forall y (K(x,y) \to (C_1(x) \to \neg C_1(y))$$

if and only if propositions

$$\begin{array}{l} K(s_1, s_1) \to (C_1(s_1) \to \neg C_1(s_1)), \\ K(s_1, s_2) \to (C_1(s_1) \to \neg C_1(s_2)), \\ K(s_2, s_1) \to (C_1(s_2) \to \neg C_1(s_1)) \;\; \text{and} \\ K(s_2, s_2) \to (C_1(s_2) \to \neg C_1(s_2)) \end{array}$$

are true in A. Because pairs $\langle s_1^{\mathcal{A}}, s_1^{\mathcal{A}} \rangle$ and $\langle s_2^{\mathcal{A}}, s_2^{\mathcal{A}} \rangle$ don't belong in interpretation $K^{\mathcal{A}}$, the predicates $K(s_1, s_1)$ and $K(s_2, s_2)$ are not true in structure $\mathcal{A}$, and by definition of truth valuation of implication the first and the last of the above propositions are true in $\mathcal{A}$. Because pair $\langle s_1^{\mathcal{A}}, s_2^{\mathcal{A}} \rangle$ is in $K^{\mathcal{A}}$, $\mathcal{A} \models K(s_1, s_2)$ and the second proposition is true in $\mathcal{A}$, if and only if $\mathcal{A} \models C_1(s_1) \to \neg C_1(s_2)$. This is true, because $s_1^{\mathcal{A}} \in C_1^{\mathcal{A}}$ and $s_2^{\mathcal{A}} \notin C_1^{\mathcal{A}}$, so $\mathcal{A} \models C_1(s_1)$ and $\mathcal{A} \models \neg C_1(s_2)$.

The third proposition is also true, because implication $C_1(s_2) \to \neg C_1(s_1)$ is true in $\mathcal{A}$, because $\mathcal{A} \not\models C_1(s_2)$. So $\mathcal{A}$ is a model for sentence $\forall x \forall y (K(x, y) \to (C_1(x) \to \neg C_1(y))$.

Because the sentences are symmetrical, $\mathcal{A}$ is also a model for sentence $\forall x \forall y (K(x, y) \to (C_2(x) \to \neg C_2(y))$. So we have shown that $\mathcal{A}$ is a model for the set of sentences.

The models will be more complex, if the colors are implemented according to formalizations (ii) or (iii).

c) We will define a structure $\mathcal{A}$ when $n = 2$, where the set of sentences is not satisfiable. We will choose as the universume $A = \{a\}$ (there is only one node) and the interpretation of predicate $K^{\mathcal{A}} = \{\langle a, a \rangle\}$. Let s be a constant symbol denoting a. Now

$$\forall x (C_1(x) \leftrightarrow \neg C_2(x)$$

is not satisfied in structure $\mathcal{A}$, if

$$\mathcal{A} \not\models C_1(s) \leftrightarrow \neg C_2(s).$$

So we can construct the interpretations of the color predicates as follows:

$$\mathcal{A} \models C_1(s) \qquad \text{and} \qquad \mathcal{A} \models C_2(s)$$

choosing

$$C_1^{\mathcal{A}} = C_2^{\mathcal{A}} = \{a\}$$

Now $\mathcal{A}$ cannot be a model for the set of sentences.