Tik-79.144 **Syksy 1999**
Logic in Computer Science: Foundations
Exercises 3
Solutions

1. We will proceed by constructing semantic tableaux for the negations of the propositions ($F(\phi)$). If all branches close (that is, there are contradictions) then ϕ is valid. If a branch is closed before the tableau is ready, then it is not necessary to continue working on that branch.

You should notice, that the semantic tableu is actually used to find models for $\neg\phi$. If all branches are contradictionary, then $\neg\phi$ doesn't have a model and its negation is valid.

b) $(A \to B) \wedge (B \to C) \to (A \to C)$:

$$
\begin{tikzcd}[column sep=small]
 & F((A \to B) \wedge (B \to C) \to (A \to C)) \arrow[d, dash] & & \\
 & T((A \to B) \wedge (B \to C)) \arrow[d, dash] & & \\
 & F(A \to C) \arrow[d, dash] & & \\
 & T(A \to B) \arrow[d, dash] & & \\
 & T(B \to C) \arrow[d, dash] & & \\
 & T(A) \arrow[d, dash] & & \\
 & F(C) \arrow[dl, dash] \arrow[dr, dash] & & \\
 \begin{array}{c} F(A) \\ \otimes \end{array} & & T(B) \arrow[dl, dash] \arrow[dr, dash] & \\
 & \begin{array}{c} F(B) \\ \otimes \end{array} & & \begin{array}{c} T(C) \\ \otimes \end{array}
\end{tikzcd}
$$

a) $A \to (B \to B)$:

- $F(A \to (B \to B))$
  - $T(A)$
    - $F(B \to B)$
      - $T(B)$
        - $F(B)$ $\otimes$

c) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$:

- $F((A \to B) \wedge (A \to C) \to (A \to B \wedge C))$
  - $T((A \to B) \wedge (A \to C))$
    - $F(A \to B \wedge C)$
      - $T(A \to B)$
        - $T(A \to C)$
          - $T(A)$
            - $F(B \wedge C)$
              - $F(A)$ $\otimes$
              - $T(B)$
                - $F(B)$ $\otimes$
                - $F(C)$
                  - $F(A)$ $\otimes$
                  - $T(C)$ $\otimes$

d) $(A \to C) \wedge (B \to C) \wedge (A \vee B) \to C$:

$$F((A \to C) \wedge (B \to C) \wedge (A \vee B) \to C)$$

$$T((A \to C) \wedge (B \to C) \wedge (A \vee B))$$

$$F(C)$$

$$T((A \to C) \wedge (B \to C))$$

$$T(A \vee B)$$

$$T(A \to C)$$

$$T(B \to C)$$

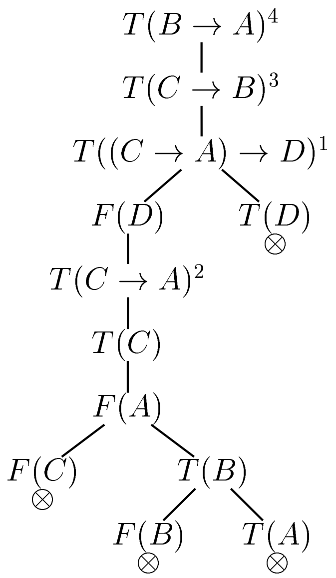
- $F(A)$
  - $F(B)$
    - $T(A)$ $\otimes$
    - $T(B)$ $\otimes$
  - $T(C)$ $\otimes$
- $T(C)$ $\otimes$

Generally it pays to construct the tree so that there are as few branches as possible.

2. When we are checking whether a proposition P is a logical consequence of a set of propositions S we put all node $T(\alpha)$ to the semantic tableaux for all $\alpha \in S$. Next we add $F(P)$ to the tableaux and use inference rules to complete it.

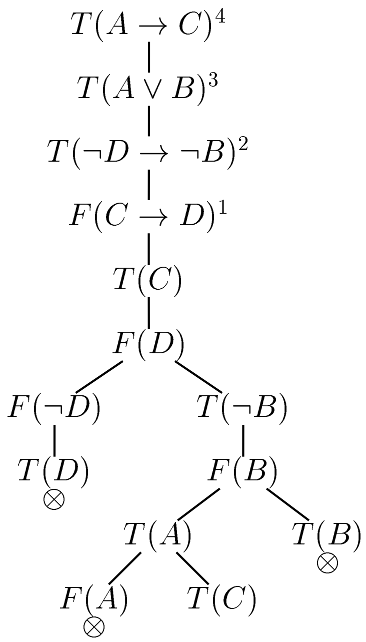
If all branches of the tableaux end in a contradiction, we know that P can't be false if all propositions in S are true and so P is a logical consequence. Otherwise, the claim doesn't hold and we can construct a counterexample from an uncontradictionary branch.

a)



As all brances closed, D is a logical consequence of the set.

b)



As there is an unclosed branch, $C \to D$ is not logical consequence of the set. We can construct a counter example from the open branch: $\mathcal{V}(A) = T$, $\mathcal{V}(B) = F$, $\mathcal{V}(C) = T$, $\mathcal{V}(D) = F$.

2. a)

$$
\begin{array}{lll}
1. & (P \to ((P \to P) \to P)) & [A1] \\
2. & ((P \to ((P \to P) \to P)) \to & \\
 & \quad ((P \to (P \to P)) \to (P \to P))) & [A2] \\
3. & ((P \to (P \to P)) \to (P \to P)) & [MP:1,2] \\
4. & (P \to (P \to P)) & [A1] \\
5. & (P \to P) & [MP:3,4]
\end{array}
$$

b)

$$
\begin{array}{lll}
1. & (Q \to R) & [P2] \\
2. & ((Q \to R) \to (P \to (Q \to R))) & [A1] \\
3. & (P \to (Q \to R)) & [MP:1,2] \\
4. & ((P \to (Q \to R)) \to ((P \to Q) \to (P \to R))) & [A2] \\
5. & ((P \to Q) \to (P \to R)) & [MP:3,4] \\
6. & (P \to Q) & [P1] \\
7. & (P \to R) & [MP:5,6]
\end{array}
$$

c)

$$
\begin{array}{lll}
1. & P & [P1] \\
2. & (Q \to (P \to R)) & [P2] \\
3. & (P \to (Q \to P)) & [A1] \\
4. & (Q \to P) & [MP:1,3] \\
5. & ((Q \to (P \to R)) \to ((Q \to P) \to (Q \to R))) & [A2] \\
6. & ((Q \to P) \to (Q \to R)) & [MP:2,5] \\
7. & (Q \to R) & [MP:4,6]
\end{array}
$$

d) We may use the above results when proving the last proposition.

$$
\begin{array}{lll}
1. & ((\neg P \to \neg\neg P) \to ((\neg P \to \neg P) \to P)) & [A3] \\
2. & (\neg P \to \neg P) & [(a)] \\
3. & ((\neg P \to \neg\neg P) \to P) & [(c):1,2] \\
4. & (\neg\neg P \to (\neg P \to \neg\neg P)) & [A1] \\
5. & (\neg\neg P \to P) & [(b):4,3]
\end{array}
$$