Tik-79.144 **Syksy 1998**
Logic in Computer Science: Foundations
Exercise 9
Solutions

1. Use unification algorithm:

a) $\sigma_0 = \epsilon$ (empty substitution)
$S_0 = \{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_0) = \{x, f(y)\}$
$\sigma_1 = \{x/f(y)\}$
$\sigma_0\sigma_1 = \{x/f(y)\}$
$S_1 = \{P(f(y), g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_1) = \{y, f(z)\}$
$\sigma_2 = \{y/f(z)\}$
$\sigma_0\sigma_1\sigma_2 = \{x/f(f(z)), y/f(z)\}$
$S_2 = \{P(f(f(z)), g(f(z)), f(a)), P(f(f(z)), g(f(z)), z)\}$
$D(S_2) = \{f(a), z\}$
$\sigma_3 = \{z/f(a)\}$
$\sigma_0\sigma_1\sigma_2\sigma_3 = \{x/f(f(f(a))), y/f(f(a)), z/f(a)\}$
$S_3 = \{P(f(f(f(a))), g(f(f(a))), f(a))\}$
The unification succeeded, the most general unifier is $\sigma_0\sigma_1\sigma_2\sigma_3$.

b) $\sigma_0 = \epsilon$
$S_0 = \{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_0) = \{x, a, y\}$
$\sigma_1 = \{x/a\}$
$S_1 = \{P(a, f(a), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_1) = \{a, y\}$
$\sigma_2 = \{y/a\}$
$S_2 = \{P(a, f(a), g(a)), P(a, f(g(a)), g(a))\}$
$D(S_2) = \{a, g(a)\}$
Terms a and $g(a)$ cannot be unified so the unification fails.

c) $\sigma_0 = \epsilon$
$S_0 = \{P(x, f(x, y)), P(y, f(y, a)), P(b, f(b, a))\}$
$D(S_0) = \{x, y, b\}$
$\sigma_1 = \{x/b\}$
$S_1 = \{P(b, f(b, y)), P(y, f(y, a)), P(b, f(b, a))\}$
$D(S_1) = \{b, y\}$
$\sigma_2 = \{y/b\}$
$S_2 = \{P(b, f(b, b)), P(b, f(b, a))\}$

$D(S_2) = \{b, a\}$
Terms b and a don't unify, so the unification fails.

d) $\sigma_0 = \epsilon$
$S_0 = \{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$
$D(S_0) = \{f(a), y, x\}$
$\sigma_1 = \{y/f(a)\}$
$S_1 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(x, f(a), f(z))\}$
$D(S_1) = \{f(a), x\}$
$\sigma_2 = \{x/f(a)\}$
$S_2 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(f(a), f(a), f(z))\}$
$D(S_2) = \{z, b, f(z)\}$ (z cannot be replaced by $f(z)$)
$\sigma_3 = \{z/b\}$
$S_3 = \{P(f(a), f(a), b), P(f(a), f(a), f(b))\}$
$D(S_3) = \{b, f(b)\}$
Terms b and $f(b)$ don't unify, so the unification fails.

2. Formalize the sentences as:

$$T(x, y) = \text{"brick } x \text{ is on brick } y\text{" and}$$
$$P(x) = \text{"brick } x \text{ is on the table".}$$

The set of premises can be formalized as follows:

$$\{\forall x\, (\exists y\, T(x, y) \to \neg P(x)),\ \forall x\, (P(x) \vee \exists y\, T(x, y)),$$
$$\forall x \forall y\, (\exists z\, T(y, z) \to \neg T(x, y))\}$$

The negation of the desired conclusion is

$$\neg \forall x \forall y\, (T(x, y) \to P(y))$$

The set of sentences can be transformed into following set of clauses

$$\{\{\neg T(x, y), \neg P(x)\},\ \{P(x), T(x, g(x))\},\ \{\neg T(y, z), \neg T(x, y)\},$$
$$\{T(c, d)\},\ \{\neg P(d)\}\}$$

Here g is a Skolem function and c and c are Skolem constants.

Next we derive contradiction from the set of clauses using binary resolution. From $\{P(x), T(x, g(x))\}$ and $\{\neg P(d)\}$ we get

$$\{T(d, g(d))\} \qquad \text{(substitution } \{x/d\})$$

Respectively, from $\{\neg T(y,z), \neg T(x,y)\}$ and $\{T(c,d)\}$ we get

$$\{\neg T(d,z)\} \qquad \text{(substitution } \{x/c, y/d\}\text{)}$$

Finally, $\{T(d, g(d))\}$ and $\{\neg T(d,z)\}$ give the empty clausul, (substitution $\{z/g(d)\}$). So the set of clauses is unsatisfiable, and the desired conclusion follows from the premises.