Tik-79.144 **Syksy 1998**
Logiikka tietotekniikassa: perusteet
Laskuharjoitus 8
Ratkaisut

1. a) Sentence: $\neg\exists x((P(x) \to P(a)) \land (P(x) \to P(b)))$:
Remove implications: $\neg\exists x((\neg P(x) \lor P(a)) \land (\neg P(x) \lor P(b)))$
Move $\neg$ inside quantifier $\exists x$:
$\forall x \neg((\neg P(x) \lor P(a)) \land (\neg P(x) \lor P(b)))$
Move negations inside subformulas:
$\forall x((P(x) \land \neg P(a)) \lor (P(x) \land \neg P(b)))$
Move $P(x)$ to the front: $\forall x(P(x) \land (\neg P(a) \lor \neg P(b)))$
Leave universal quantifiers out: $P(x) \land (\neg P(a) \lor \neg P(b))$
Form the clauses: $\{\{P(x)\},\ \{\neg P(a),\ \neg P(b)\}\}$

b) Sentence: $\forall y \exists x P(x, y)$:
Skolemization: $\forall y P(f(y), y)$
Leave universal quantifiers out: $P(f(y), y)$
Form the clauses: $\{\{P(f(y), y)\}\}$

c) Sentence: $\neg\forall y \exists x G(x, y)$:
Move $\neg$ after the quantifiers: $\exists y \forall x \neg G(x, y)$
Skolemization: $\forall x \neg G(x, c)$
Leave universal quantifiers out: $\neg G(x, c)$
Form the clauses: $\{\{\neg G(x, c)\}\}$

d) Sentence: $\exists x \forall y \exists z(P(x, z) \lor P(z, y) \to G(x, y))$:
Remove the implication: $\exists x \forall y \exists z(\neg(P(x, z) \lor P(z, y)) \lor G(x, y))$
Move negations inside:
$\exists x \forall y \exists z((\neg P(x, z) \land \neg P(z, y)) \lor G(x, y))$
Construct the prenex-form:
$\exists x \forall y \exists z((\neg P(x, z) \lor G(x, y)) \land (\neg P(z, y) \lor G(x, y)))$
Skolemization: $\forall y \exists z((\neg P(c, z) \lor G(c, y)) \land (\neg P(z, y) \lor G(c, y)))$
Skolemization: $\forall y((\neg P(c, f(y)) \lor G(c, y)) \land (\neg P(f(y), y) \lor G(c, y)))$
Leave universal quantifiers out:
$(\neg P(c, f(y)) \lor G(c, y)) \land (\neg P(f(y), y) \lor G(c, y))$
Form the clauses:
$\{\{\neg P(c, f(y)),\ G(c, y)\},\ \{\neg P(f(y), y),\ G(c, y)\}\}$

2. The Herbrand Universume U of a set of clauses S is formed by terms which can be constructed using the constant and function symbols in S. If there is no constant symbols in S, we add an arbitary constant (e.g. a) to the set. The Herbrand Base B is the set of atomic sentences that can be constructed using the predicate symbols in S and the terms in U.

a) $U = \{c\}$, $B = \{G(c,c)\}$.

b) $U = \{a, f(a), f(f(a)), ...\}$, $B = \{P(e_1, e_2) \mid e_1 \in U, e_2 \in U\}$.

c) $U = \{a, b\}$, $B = \{P(a), P(b)\}$.

d) $U = \{a\}$, $B = \{P(a,a), G(a,a)\}$.

e) $U = \{a, b, f(a), f(b), f(f(a)), f(f(b)), ...\}$,
$B = \{P(e_1, e_2) \mid e_1 \in U, e_2 \in U\} \cup \{Q(e_1, e_2) \mid e_1 \in U, e_2 \in U\}$.

f) $U = \{a, f(a,a), f(a, f(a,a)), f(f(a,a),a), f(f(a,a), f(a,a)), ...\}$,
$B = \{P(e) \mid e \in U\} \cup \{Q(e) \mid e \in U\}$.

3. First transform the set of sentences to clausul form:

$$\{\forall x P(x,a,x),\ \neg\exists x \exists y \exists z (P(x,y,z) \wedge \neg P(x, f(y), f(z)))\}$$

After transforming the sentences to prenex-form and skolemization the set of clauses is

$$\{\{P(x,a,x)\},\ \{\neg P(x,y,z),\ P(x, f(y), f(z))\}\}$$

We define notation $f^i(a)$ for each $i \in \mathbb{N}$: $f^0(a)$ denotes a, $f^1(a)$ denotes $f(a)$, $f^2(a)$ denotes $f(f(a))$ and so on. The Herbrand Universume of the set is now:

$$U = \{a, f(a), f(f(a)), ...\} = \{f^i(a) \mid i \in \mathbb{N}\}$$

and the Herbrand Base

$$B = \{P(f^i(a), f^j(a), f^k(a)) \mid i \in \mathbb{N}, j \in \mathbb{N}, k \in \mathbb{N}\}$$

We have to find a minimal model M for the set of clauses. The model will be a subset of B.

If we want to satisfy clause $\{P(x,a,x)\}$ with all $x \in U$, we have to add all atomic sentences of the form $P(f^i(a), a, f^i(a))$ $(i \in \mathbb{N})$ to the model. When these sentences are true their negations are not true, and to satisfy the second clause $\{\neg P(x,y,z), P(x,f(y),f(z))\}$ when $x = f^i(a), y = a, z = f^i(a)$, we have to add also all sentences of the form $P(f^i(a), f(a), f^{i+1}(a))$ $(i \in \mathbb{N})$ to the model M.

Now all the sentences of the form $\neg P(f^i(a), f(a), f^{i+1}(a))$ $(i \in \mathbb{N})$ are not true in M, and we have to add more sentences to the model, namely all sentences of the form $P(f^i(a), f^2(a), f^{i+2}(a))$.

By continuing this procedure we find out that the clause $\{\neg P(x,y,z), P(x,f(y),f(z))\}$ can be satisfied only by adding all the sentences of the form $P(f^i(a), f^j(a), f^{i+j}(a))$ $(i \in \mathbb{N}, j \in \mathbb{N})$ to the model M. Now the model is complete.

If a is interpreted as the natural number 0 and f is interpreted as the successor function $s : \mathbb{N} \to \mathbb{N} : s(x) = x + 1$, the interpretation of predicate P (in M) is a trinary predicate that is true if and only if the third argument is the sum of the two first arguments. Thus original set of sentences models the summation of natural numbers (in M).