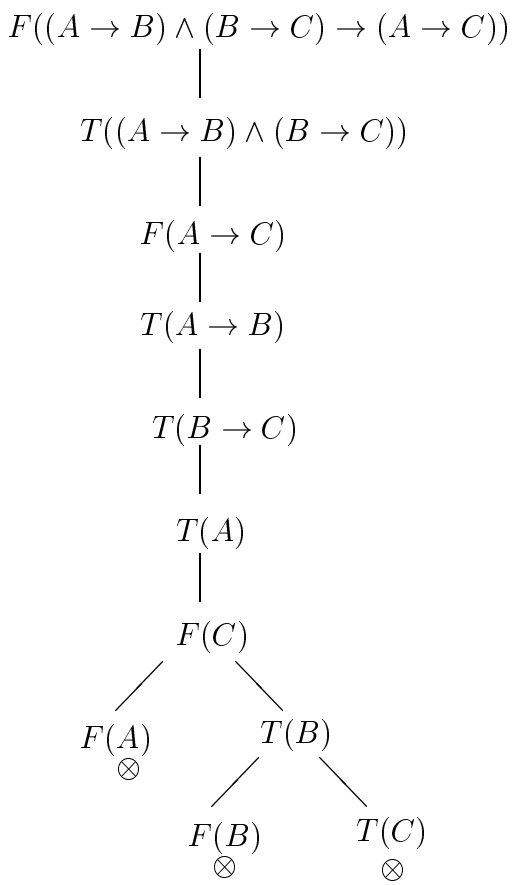
Tik-79.144 **Syksy 1998**
Logic in Computer Science: Foundations
Exercises 3
Solutions

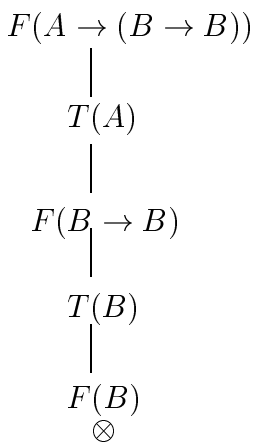
1. We will proceed by constructing semantic tableaux for the negations of the propositions ($F(\phi)$). If all branches close (that is, there are contradictions) then ϕ is valid. If a branch is closed before the tableau is ready, then it is not necessary to continue working on that branch.

You should notice, that the semantic tableu is actually used to find models for $\neg\phi$. If all branches are contradictionary, then $\neg\phi$ doesn't have a model and its negation is valid.

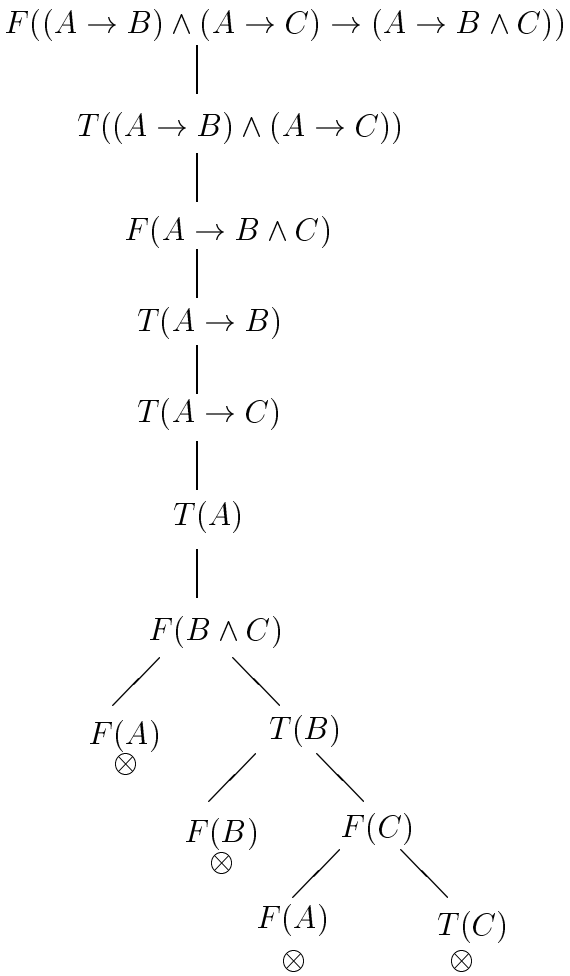
b) $(A \to B) \wedge (B \to C) \to (A \to C)$:



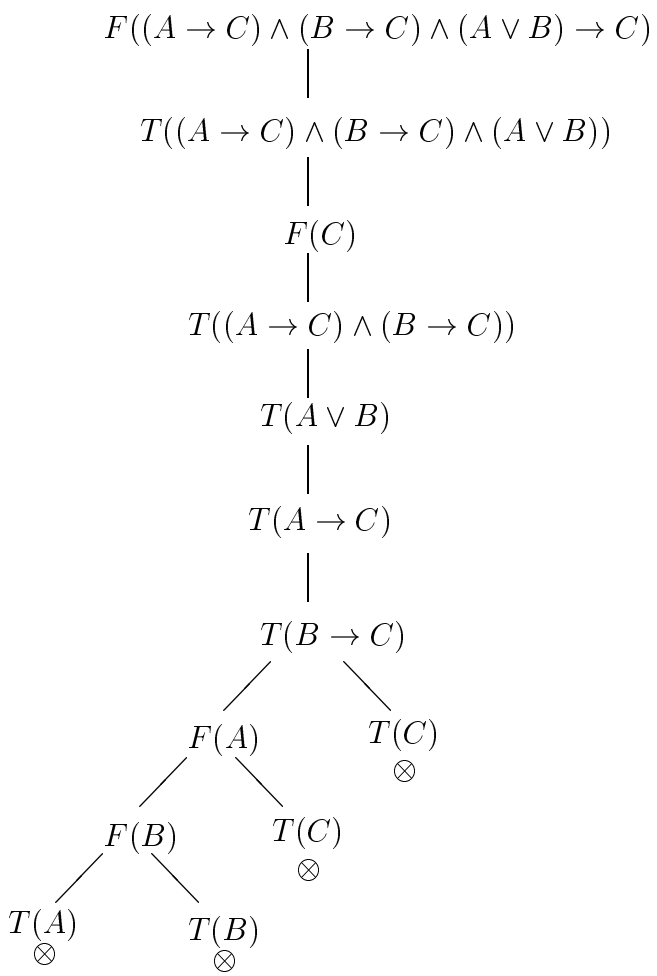
a) $A \to (B \to B)$:



c) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$:



d) $(A \to C) \wedge (B \to C) \wedge (A \vee B) \to C$:



Generally it pays to construct the tree so that there are as few branches as possible.

2. a) First equivalences and implications are removed from the proposition, and then the negations are transferred to the "innermost" level using De Morgan rules.

$$
\begin{aligned}
&\neg((A \leftrightarrow \neg B) \to C) \\
&\qquad \equiv \neg((A \to \neg B) \wedge (\neg B \to A) \to C) && [\leftrightarrow e] \\
&\qquad \equiv \neg(\neg((\neg A \vee \neg B) \wedge (\neg\neg B \vee A)) \vee C) && [\to e] \\
&\qquad \equiv (\neg A \vee \neg B) \wedge (A \vee B) \wedge \neg C \ (*) && [\neg\neg e, DM]
\end{aligned}
$$

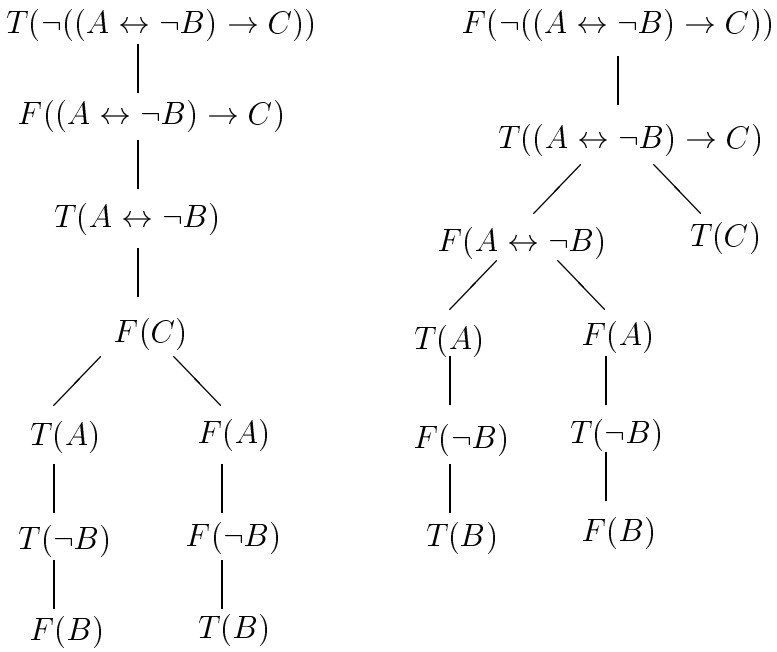
Now the conjunctive normal form is found. The disjunctive normal form is found by using the distributivity of conjunction over

distribution (ie. $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$).

$$\begin{aligned}(*) &\equiv (\neg A \vee \neg B) \wedge ((B \wedge \neg C) \vee (A \wedge \neg C)) \\ &\equiv (\neg A \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) \vee \\ &\quad (\neg B \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) \\ &\equiv (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge A \wedge \neg C) \vee \\ &\quad (\neg B \wedge B \wedge \neg C) \vee (\neg B \wedge A \wedge \neg C) \\ &\equiv (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)\end{aligned}$$

Now the disjunctive normal form is found. In the last step unsatisfiable conjunctions were left out.

b) It is easy to find all the models of a proposition by using its semantic tableau, and the disjunctive normal form can be constructed from these models. We'll construct semantic tableaux for both the proposition and its negation:

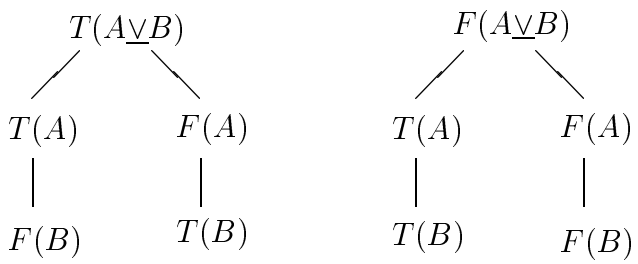


The disjunctive normal form can be read from the *open* brances of the first tableau: $(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$. There is one conjunction for each open branch. If there is a node $T(A)$ in the branch, then the proposition A will be in the conjunction, and if there is a node $F(A)$, proposition $\neg A$ is used.

From the other tableau a disjunctive normal form can be found for the negation of the original proposition: $(A \wedge B) \vee (\neg A \wedge$

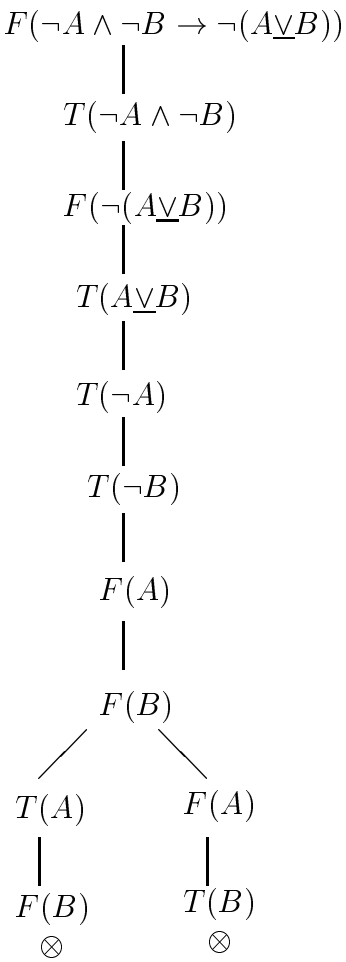
$\neg B) \vee C$. By taking the negation $\neg((A \wedge B) \vee (\neg A \wedge \neg B) \vee C)$ we have a proposition that is equivalent with the original proposition. Now we can use the De Morgan rules to construct the conjunctive normal form for the original proposition: $(\neg A \vee \neg B) \wedge (A \vee B) \wedge \neg C$. You should check, that the results agrees with results from (a).

3. Because the exclusive disjunction $A \underline{\vee} B$ is equivalent to proposition $A \leftrightarrow \neg B$ the atomic tableaux should be defined as follows:



In other words, $A \underline{\vee} B$ is true if and only if either A or B is true, not both. Respectively, $A \underline{\vee} B$ is false if and only if both A and B have same truth values.

The proposition $\neg A \wedge \neg B \rightarrow \neg(A \underline{\vee} B)$ can be proved with following tableau:


$F(\neg A \wedge \neg B \to \neg(A \underline{\vee} B))$
$T(\neg A \wedge \neg B)$
$F(\neg(A \underline{\vee} B))$
$T(A \underline{\vee} B)$
$T(\neg A)$
$T(\neg B)$
$F(A)$
$F(B)$
$T(A)$
$F(A)$
$F(B)$
$\otimes$
$T(B)$
$\otimes$