Tik-79.144 **Syksy 1998**
Logic in Computer Science: Foundations
Exercises 2
Solutions

1. a) We will use atomic propositions $P1$, $K1$ and $V1$ to denote respectively that the lamp post 1 has red, yellow and green light on (the letters come from the initial letters of the colors in Finnish). Let $P2$, $K2$ and $V2$ be the corresponding propositions for lamp post 2. Now we'll go through each requirement and present the set of propositions that correspond to the requirement.

(i) For lamp post 1 we need proposition $P1 \vee K1 \vee V1$ (at least one lamp is alight) and propositions $P1 \rightarrow \neg K1 \wedge \neg V1$, $K1 \rightarrow \neg P1 \wedge \neg V1$, $V1 \rightarrow \neg P1 \wedge \neg K1$ (at most one lamp is alight). Also, corresponding propositions are needed for lamp post 2.

(ii) The needed proposition is $\neg(V1 \wedge V2)$.

(iii) We need propositions $P1 \rightarrow (K2 \vee V2)$ and $P2 \rightarrow (K1 \vee V1)$.

b) The claim "the yellow light is alight on both traffic lights" translates into proposition $K1 \wedge K2$. Let $\mathcal{V}$ be a valuation, than maps atomic propositions $K1$ and $K2$ to true (T) and other atomic propositions to false (E). Now $\mathcal{V}(K1 \wedge K2) = T$, because $\mathcal{V}(K1) = T$ and $\mathcal{V}(K2) = T$. Also, every for every proposition α in the point (a) it holds that $\alpha = T$ (you should check this by yourself). So $\mathcal{V}$ is a model of the set of propositions, where $K1 \wedge K2$ is true. You can use semantic tableaux to find models.

c) Let $\mathcal{V}$ be a valuation which maps atomic propositions $V1$ and $V2$ to true, and rest atomic propositions to false. Now $\mathcal{V}(\neg(V1 \wedge V2)) = E$, so the set of propositions is not satisfied.

d) The requirements are not sufficient, because in real life red and yellow lights may be on at the same time. It is possible to lighten the conditions of (i) to allow this (think how this may be done by yourself). A worse problem is that the propositions don't specify the working order of the lights (e.g. that the yellow light should follow the green one). It is quite difficult to model this kind of behaviour with propositional logic.

2. a) Let's construct a truth table for the above set of propositions. We'll use a shorthand notation α_i for propositions $(Pi \vee Ki \vee Vi) \wedge$

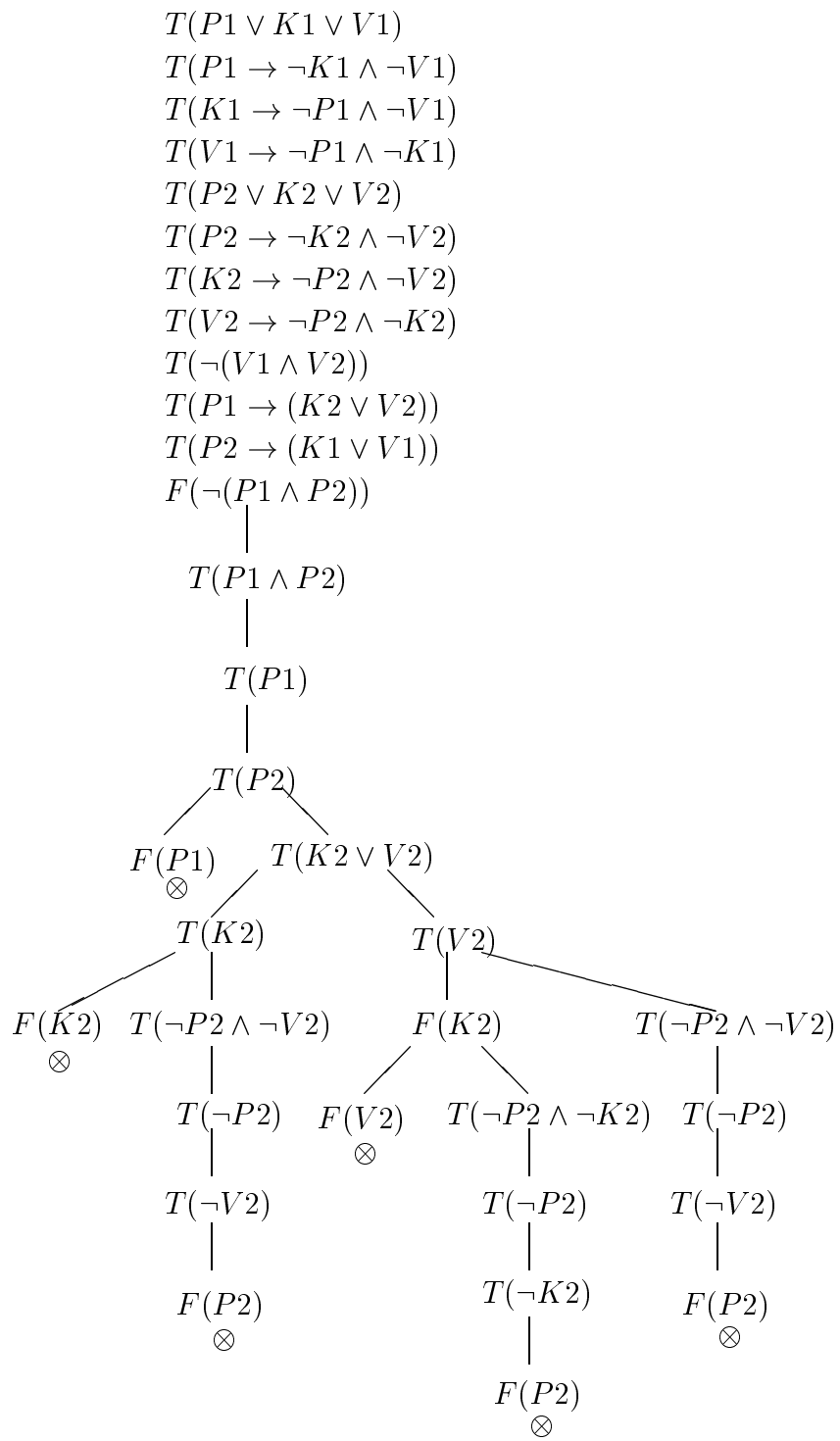
$(Pi \rightarrow \neg Ki \wedge \neg Vi) \wedge (Ki \rightarrow \neg Pi \wedge \neg Vi) \wedge (Vi \rightarrow \neg Pi \wedge \neg Ki)$ (which means that the lamp post i has exactly one light on). The rows marked with stars are models of the set of propositions.

P1	K1	V1	P2	K2	V2	α_1	α_2	$\neg(V1 \wedge V2)$	$P1 \rightarrow (K2 \vee V2)$	$P2 \rightarrow (K1 \vee V1)$	
F	F	F	F	F	F	F	F	T	T	T	
F	F	F	F	F	T	F	T	T	T	T	
F	F	F	F	T	F	F	T	T	T	T	
F	F	F	F	T	T	F	F	T	T	T	
F	F	F	T	F	F	F	T	T	T	F	
F	F	F	T	F	T	F	F	T	T	F	
F	F	F	T	T	F	F	F	T	T	F	
F	F	F	T	T	T	F	F	T	T	F	
F	F	T	F	F	F	T	F	T	T	T	
F	F	T	F	F	T	T	T	F	T	T	
F	F	T	F	T	F	T	T	T	T	T	*
F	F	T	F	T	T	T	F	F	T	T	
F	F	T	T	F	F	T	T	T	T	T	*
F	F	T	T	F	T	T	F	F	T	T	
F	F	T	T	T	F	T	F	T	T	T	
F	F	T	T	T	T	T	F	F	T	T	
F	T	F	F	F	F	T	F	T	T	T	
F	T	F	F	F	T	T	T	T	T	T	*
F	T	F	F	T	F	T	T	T	T	T	*
F	T	F	F	T	T	T	F	T	T	T	
F	T	F	T	F	F	T	T	T	T	T	*
F	T	F	T	F	T	T	F	T	T	T	
F	T	F	T	T	F	T	F	T	T	T	
F	T	F	T	T	T	T	F	T	T	T	

P1	K1	V1	P2	K2	V2	α_1	α_2	$\neg(V1 \wedge V2)$	$P1 \rightarrow (K2 \vee V2)$	$P2 \rightarrow (K1 \vee V1)$	
F	T	T	F	F	F	F	F	T	T	T	
F	T	T	F	F	T	F	T	F	T	T	
F	T	T	F	T	F	F	T	T	T	T	
F	T	T	F	T	T	F	F	F	T	T	
F	T	T	T	F	F	F	T	T	T	T	
F	T	T	T	F	T	F	F	F	T	T	
F	T	T	T	T	F	F	F	T	T	T	
F	T	T	T	T	T	F	F	F	T	T	
T	F	F	F	F	F	T	F	T	F	T	
T	F	F	F	F	T	T	T	T	T	T	*
T	F	F	F	T	F	T	T	T	T	T	*
T	F	F	F	T	T	T	F	T	T	T	
T	F	F	T	F	F	T	T	T	F	F	
T	F	F	T	F	T	T	F	T	T	F	
T	F	F	T	T	F	T	F	T	T	F	
T	F	F	T	T	T	T	F	T	T	F	
T	F	T	F	F	F	F	F	T	F	T	
T	F	T	F	F	T	F	T	F	T	T	
T	F	T	F	T	F	F	T	T	T	T	
T	F	T	F	T	T	F	F	F	T	T	
T	F	T	T	F	F	F	T	T	F	T	
T	F	T	T	F	T	F	F	F	T	T	
T	F	T	T	T	F	F	F	T	T	T	
T	F	T	T	T	T	F	F	F	T	T	
T	T	F	F	F	F	F	F	T	F	T	
T	T	F	F	F	T	F	T	T	T	T	
T	T	F	F	T	F	F	T	T	T	T	
T	T	F	F	T	T	F	F	T	T	T	
T	T	F	T	F	F	F	T	T	F	T	
T	T	F	T	F	T	F	F	T	T	T	
T	T	F	T	T	F	F	F	T	T	T	
T	T	F	T	T	T	F	F	T	T	T	
T	T	T	F	F	F	F	F	T	F	T	
T	T	T	F	F	T	F	T	F	T	T	
T	T	T	F	T	F	F	T	T	T	T	
T	T	T	F	T	T	F	F	F	T	T	
T	T	T	T	F	F	F	T	T	F	T	
T	T	T	T	F	T	F	F	F	T	T	
T	T	T	T	T	F	F	F	T	T	T	
T	T	T	T	T	T	F	F	F	T	T	

There are seven models (out of $2^6 = 64$ valuations). The claim "both red lights are not on at the same time" can be formalized as $\neg(P1 \wedge P2)$. Examining the models we can see that the proposition $\neg(P1 \wedge P2)$ is true in each of them (check it), so it is a logical consequence of the set of propositions.

b) The same exercise can be done more compactly using semantic tableaux. The desired consequence $\neg(P1 \wedge P2)$ is initially marked as false, because we want to prove that there is a contradiction if the proposition is not true. Because all branches of the tableau close, $\Sigma \cup \{\neg(\neg(P1 \wedge P2))\}$ is unsatisfiable (Σ is the set of the propositions) and so $\neg(P1 \wedge P2)$ follows logically from Σ.



The moral of the exercise is that truth tables are not the best way to find models if there are many atomic propositions.

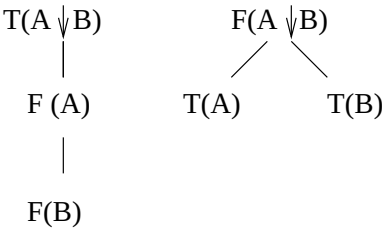
3. a) Negation

$$\begin{aligned} \neg A &\equiv \neg A \wedge \neg A \\ &\equiv (A \downarrow A) \end{aligned}$$

b) Disjunction

$$\begin{aligned} A \vee B &\equiv \neg(\neg A \wedge \neg B) \\ &\equiv \neg(A \downarrow B) \\ &\equiv ((A \downarrow B) \downarrow (A \downarrow B)) \end{aligned}$$

c) Conjunction

$$\begin{aligned} A \wedge B &\equiv \neg\neg A \wedge \neg\neg B \\ &\equiv (\neg A \downarrow \neg B) \\ &\equiv ((A \downarrow A) \downarrow (B \downarrow B)) \end{aligned}$$

The atomic tableaux can be constructed by starting with proposition $(\neg A \wedge \neg B)$ and checking what is the resulting tableau like.