T-79.1001/2 **Syksy 2007**
Introduction to Theoretical Computer Science (T/Y)
Session 1
Answers to demonstration exercises

4. **Problem**: Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ by the rule:

$$(m, n) \sim (p, q) \quad \Leftrightarrow \quad m + n = p + q.$$

Prove that this is an equivalence relation, and describe intuitively ("geometrically") the equivalence classes it determines.

Solution: The relation $\backsim \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ is defined in the following way:

$$(m, n) \sim (p, q) \quad \Leftrightarrow \quad m + n = p + q$$

In other words, two pairs are equivalent when their sums are the same.

A relation is an equivalence relation when it is symmetric, transitive and reflexive.

i) The relation $\backsim$ is symmetric, if $(m, n) \sim (p, q)$ always when $(p, q) \sim (m, n)$. Because

$$m + n = p + q \Leftrightarrow p + q = m + n,$$

$((p, q), (m, n))$ is always in the relation when $((m, n), (p, q))$ is. Thus the relation is symmetric.

ii) The relation $\backsim$ is reflexive, if for all $(m, n) \in \mathbb{N}$ holds that $(m, n) \sim (m, n)$. Since

$$m + n = m + n,$$

the condition is fulfilled.

iii) The relation $\backsim$ is transitive, if always when $(m, n) \sim (p, q)$ and $(p, q) \sim (k, l)$, also $(m, n) \sim (k, l)$.
Given

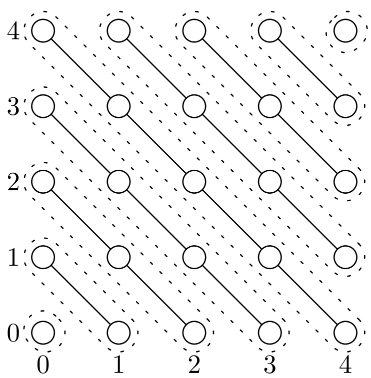
$$m + n = p + q \wedge p + q = k + l,$$

then

$$m + n = p + q = k + l \Rightarrow m + n = k + l,$$

and thus the relation is also transitive.

Because all three conditions hold, $\backsim$ is an equivalence relation. Below, the first elements of the relation as a graph.



From the figure it can be seen that the equivalence classes defined by the relation correspond with the lines that are parallel to the line $y = -x$.

5. **Problem**:Prove by induction that if X is a finite set of cardinality $n = |X|$, then its power set $\mathcal{P}(X)$ is of cardinality $|\mathcal{P}(X)| = 2^n$.

   Solution: Base case: $X = \emptyset$. Then $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $|\mathcal{P}(\emptyset)| = 1 = 2^0$.

   Induction hypothesis: we assume there exists a $k \in \mathbb{N}$ such that formula holds for all $n \leq k$.

   Inductive step: let $|X| = k + 1$. Denote $X = Y \cup \{x\}$. By the induction hypothesis $|\mathcal{P}(Y)| = 2^k$. The set $\mathcal{P}(X)$ contains all elements of $\mathcal{P}(Y)$ and the union of the elements of $\mathcal{P}(Y)$ with $\{x\}$. Thus we get $|\mathcal{P}(X)| = 2 \cdot 2^k = 2^{k+1}$.

6. **Problem**: Prove by induction that every partial order defined on a finite set S contains at least one minimal element. Furthermore, provide examples showing that the minimal element is not necessarily unique (i.e. there can be more than one), and that in an infinite set S the claim does not necessarily hold.

   Solution: We apply induction w.r.t. the size of S .

   1° Base case: Consider the smallest possible non-empty set $S_1 = \{a_1\}$. This set has only one possible partial order $R_1 = \{(a_1, a_1)\}$. (A partial order is a reflexive, anti-symmetric and transitive binary relation).
   An element $a \in S$ is a minimal element exactly when it does not appear on the right side of any pair (except for the reflexive arc). More formally:

   $$\forall a, b \in S : (b, a) \in R \Rightarrow a = b,$$

   In the partial order R_1 the element a_1 fulfils the condition above and it is thus the minimal element.

   2° Induction hypothesis: assume there exists a natural number $n > 1$ such that when $|S| < n$, all partial orders formed from the elements of S have a minimal element.

   3° Inductive step: let $S_n = \{a_1, \ldots, a_n\}$ be a set with n elements and let R_n be any partial order formed from the elements of S_n. Choose an arbitrary element a_i ($1 \leq i \leq n$), remove it from S_n, and also remove all pair which refer to it from the relation:

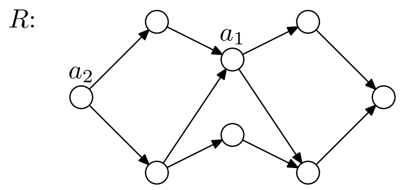
   $$S'_n = S_n - \{a_i\}$$
   $$R'_n = \{(a, b) \in R_n \mid a \neq a_i \wedge b \neq a_i\}$$

   Now R'_n is also a partial order (this follows from the transitivity of R_n). Because the set S'_n contains $n - 1$ elements ($< n$), R'_n has by the induction hypothesis at least one minimal element, which we denote $a_{\min}$.
   Consider R_n again. There are two possible cases:

   i) If the arc $(a_i, a_{\min}) \notin R_n$, then $a_{\min}$ is also the minimal element of R_n, because the only difference between R_n and R'_n is the element a_i and the arcs attached it.

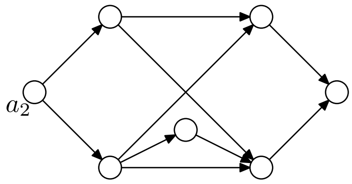
   ii) If the arc $(a_i, a_{\min}) \in R_n$, $a_{\min}$ cannot be a minimal element. Because $a_{\min}$ is the minimal element of R'_n and because a partial order is always transitive, the relation R_n cannot have the arc $(b, a_i) \in R_n, b \neq a_i$. Otherwise also the arc $(b, a_{\min}) \in R'_n$, and not $a_{\min}$ would be the minimal element of R'_n. Thus a_i is a minimal element of R_n and the proof is complete.

The induction step of the proof can be visualised by inspecting the the following partial order (the reflexive and the transitive arcs have been left out for the sake of clarity):



We remove element a_1. We obtain the partial order R':

R':



The minimal element of this partial order is a_2. Because the original order does not contain the arc (a_1, a_2), we see that a_2 is also the minimal element of R. This corresponds to the first case (i) of the inductive step. Case (ii) is obtained by removing a_2.

A partial order defined over an infinite domain does not necessarily have a minimal element. One example is the set of natural numbers $\mathbb{Z}$ and the order $\leq$.

A simple example of a partial order with several minimal elements is

$$R = \{(p,p), (q,q), (l,l), (q,l), (p,l)\}.$$

Both p and q are minimal elements.