4 **Problem**: Prove that the following *de Morgan formulas* hold for all sets \( A, B \subseteq U \):

\[
A \cup B = \overline{A \cap \overline{B}}, \quad A \cap \overline{B} = \overline{A \cup B}
\]

**Answer**: Two sets are equal when they have the same elements. Let us first examine the set \( A \cup B \). Let \( a \in A \cup B \). Then, \( a \notin A \cup B \). By the definition of union this means that \( a \notin A \) and \( a \notin B \). This means that \( a \in \overline{A} \) and \( a \in \overline{B} \). Since every step in the proof preserves equivalence, the proof applies also for the other direction.

Next, consider \( a \in A \cap B \). Then, \( a \notin A \cap B \) so \( a \notin A \) or \( a \notin B \). Now \( a \in \overline{A} \) or \( a \in \overline{B} \) so by definition of the union \( a \in \overline{A} \cup \overline{B} \).

5. **Problem**: Define a relation \( \sim \) on the set \( \mathbb{N} \times \mathbb{N} \) by the rule:

\[
(m, n) \sim (p, q) \iff m + n = p + q.
\]

Prove that this is an equivalence relation, and describe intuitively (*geometrically*) the equivalence classes it determines.

**Solution**: The relation \( \sim \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \) is defined in the following way:

\[
(m, n) \sim (p, q) \iff m + n = p + q
\]

In other words, two pairs are equivalent when their sums are the same.

A relation is an equivalence relation when it is symmetric, transitive and reflexive.

i) The relation \( \sim \) is symmetric, if \((m, n) \sim (p, q)\) always then \((p, q) \sim (m, n)\). Because

\[
m + n = p + q \iff p + q = m + n,
\]

\((p, q), (m, n)\) is always in the relation when \((m, n), (p, q)\) is. Thus the relation is symmetric.

ii) The relation \( \sim \) is reflexive, if for all \((m, n) \in \mathbb{N}\) holds that \((m, n) \sim (m, n)\). Since

\[
m + n = m + n,
\]

the condition is fulfilled.

iii) The relation \( \sim \) is transitive, if always when \((m, n) \sim (p, q)\) and \((p, q) \sim (k, l)\), also \((m, n) \sim (k, l)\).

Given

\[
m + n = p + q \land p + q = k + l,
\]

then

\[
m + n = p + q = k + l \Rightarrow m + n = k + l,
\]

and thus the relation is also transitive.

Because all three conditions hold, \( \sim \) is an equivalence relation. Below, the first elements of the relation as a graph.
From the figure it can be seen that the equivalence classes defined by the relation correspond with the lines that are parallel to the line $y = -x$.

6. **Problem:** Prove by induction that if $X$ is a finite set of cardinality $n = |X|$, then its power set $P(X)$ is of cardinality $|P(X)| = 2^n$.

**Solution:** Base case: $X = \emptyset$. Then $P(\emptyset) = \{\emptyset\}$ and $|P(\emptyset)| = 1 = 2^0$.

Induction hypothesis: we assume there exists a $k \in \mathbb{N}$ such that formula holds for all $n \leq k$.

Inductive step: let $|X| = k + 1$. Denote $X = Y \cup \{x\}$. By the induction hypothesis $|P(Y)| = 2^k$. The set $P(X)$ contains all elements of $P(Y)$ and the union of the elements of $P(Y)$ with $\{x\}$. Thus we get $|P(X)| = 2 \cdot 2^k = 2^{k+1}$. 