T-79.1001/1002 **Spring 2006**
Introduction to Theoretical Computer Science T/Y
Tutorial 1, 24–25 January
Problems

Remember to enroll for the course using the TOPI registration system by 27 Jan, 6 p.m. For bookkeeping reasons, registration is **compulsory**, *even if you were not intending to attend the lectures or the tutorial sessions.*

Homework problems:

1. (a) Let $A = \{a, b, c, d\}$, and define a relation $R \subseteq A \times A$ as follows:

$$R = \{(a, b), (b, c), (b, d), (c, a), (d, d)\}.$$

Draw the graphs corresponding to the following relations:

$$\text{(a)}\ R, \qquad \text{(b)}\ R^{-1}, \qquad \text{(c)}\ R \circ R, \qquad \text{(d)}\ (R \circ R) - R^{-1}.$$

Are some of these relations actually functions?

(b) List all the equivalence relations (partitions) on the set $\{a, b, c\}$.

2. Let $\Sigma = \{a, b\}$. Give some examples of strings from each of the following languages (at least three strings per language):

(a) $\{w \in \Sigma^* \mid w$ the number of a's in w is even and the number of b's is divisible of three$\}$;
(b) $\{a^{2n}b^{3m} \mid n, m \geq 0\}$;
(c) $\{uvu^Rv^R \mid u, v \in \Sigma^*\}$;[1]
(d) $\{w \in \Sigma^* \mid \exists u, v \in \Sigma^* \text{ s.t. } w = uu = vvv\}$.

3. The *reversal* of a string $w \in \Sigma^*$, denoted w^R, is defined inductively by the rules:

(i) $\varepsilon^R = \varepsilon$;
(ii) if $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$, then $w^R = au^R$.

It was proved in class (cf. also Lewis & Papadimitriou, p. 43) that for any strings $u, v \in \Sigma^*$ it is the case that $(uv)^R = v^Ru^R$. Prove in a similar manner, by induction based on the above definition of reversal, the following facts:

(a) $(w^R)^R = w$;
(b) $(w^k)^R = (w^R)^k$, for any $k \geq 0$.

Demonstration problems:

4. Let A and B be subsets of a given fundamental set U. Prove the correctness of the following *de Morgan formulas* that relate the unions, intersections, and complements of A and B to each other:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

5. Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ by the rule:

$$(m, n) \sim (p, q) \quad \Leftrightarrow \quad m + n = p + q.$$

Prove that this is an equivalence relation, and describe intuitively ("geometrically") the equivalence classes it determines.

6. Prove by induction that if X is a finite set of cardinality $n = |X|$, then its power set $\mathcal{P}(X)$ is of cardinality $|\mathcal{P}(X)| = 2^n$.

[1] For a definition of the notation w^R see Problem 2.