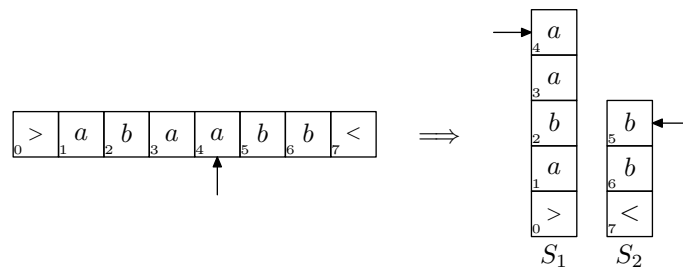


4. **Problem:** Show that pushdown automata with two stacks (rather than just one as permitted by the standard definition) would be capable of recognizing exactly the same languages as Turing machines.

Solution: We first show that a two-stack pushdown automaton can simulate a Turing machine. The only difficulty is to find a way to simulate the Turing machine tape using two stacks. This can be done using a construction that is similar to the one presented in the first problem: the first stack holds the part of tape that is left to the read/write head (in reversed order), and the second stack holds the symbols that are right to the head.



The computation of the automaton can be divided into two parts:

- Initialization, when the automaton copies the input to stack S_1 one symbol at a time, and then moves it, again one-by-one, to stack S_2 . (With the exception of the first symbol).
- Simulation, where the automaton decides its next transition by examining the top symbol of S_1 . If the machine moves its head to left, the top element of S_1 is moved into S_2 . If it moves to the other direction, the top element of S_2 is moved to S_1 .

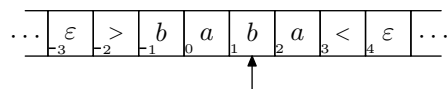
A two-stack pushdown automaton that is formed using these principles simulates a given Turing machine. The formal details are presented in an appendix.

Next we show that we can simulate a two-stack pushdown automaton using a Turing machine. This can be done trivially using a two tape nondeterministic Turing machine where both stacks are stored on their own tapes.

5. **Problem:** Extend the notion of a Turing machine by providing the possibility of a two-way infinite tape. Show that nevertheless such machines recognize exactly the same languages as the standard machines whose tape is only one-way infinite.

Solution: A Turing machine with a two-way infinite tape works otherwise in a same way than a standard machine except that the position of the tape start symbol ($>$) is not fixed and it can move in a same way than the end symbol ($<$). The tape positions are indexed by the set \mathbb{Z} of integers where 0 denotes the initial position of $>$.

We can simulate such a Turing machine by a two-track one-way Turing machine. Conceptually, we divide the tape into two parts: upper and lower. The upper part holds the two-way tape cells $i \geq 0$ and the lower part cells $i < 0$. For example, a two-way tape:



enumeration can not be done parallel to the x -axis; when doing this all indices would be used to enumerate only the y -axis and no pair $(x, y), y > 0$ would ever be reached.

The enumerating scheme above can be defined as follows:

$$f(x, y) = x + \sum_{k=1}^{x+y} k = x + \frac{(x+y)(x+y+1)}{2}$$

For an example, $f(3, 1) = 13$, that is, the running number of pair $(3, 1)$ is 13. The function $f(x, y)$ is a bijection; for every running number there exists a unambiguous pair of numbers. Calculating a coordinate from a given index is relatively difficult, and is discussed in the appendix at the end of these solutions.

The set of positive rational numbers \mathbb{Q}^+ can be presented as a pair of numbers $\mathbb{N} \times \mathbb{N}$ by $(x, y) \equiv \frac{x}{y}, y \neq 0$. This is a proper subset of the countably infinite set $\mathbb{N} \times \mathbb{N}$. By Problem 3, \mathbb{Q}^+ is either finite or countably infinite. If \mathbb{Q}^+ was finite, there should exist some rational number $\frac{x}{y}, x \in \mathbb{N}, y \in \mathbb{N}, y \neq 0$, that would have the greatest running number $n < \infty$ (in the enumeration of \mathbb{Q}). This cannot be, because using the figure above one could always find a rational number that would have a running number $n' > n$. Hence, we have contradiction with the assumption that \mathbb{Q}^+ is finite. Therefore \mathbb{Q}^+ is countably infinite. By the same argument, the set \mathbb{Q}^- :

$$\mathbb{Q}^- = \{(-x, y) \mid (x, y) \in \mathbb{Q}^+\}$$

is countably infinite. Thus, the set $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$ is the union of two countably infinite sets, and it too is countably infinite.

Appendix: the formalisation of solution 5

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ be a two-way tape Turing machine. Define a standard Turing machine M' as follows:

$$\begin{aligned} M' &= (Q', \Sigma', \Gamma', \delta', q_0, q_{\text{acc}}, q_{\text{rej}}) \\ Q' &= Q \cup \{q' \mid q \in Q\} \\ \Sigma' &= (\Sigma \cup \{<', >'\}) \times (\Sigma \cup \{<', >'\}) \\ \Gamma' &= (\Gamma \cup \{<', >'\}) \times (\Gamma \cup \{<', >'\}) \end{aligned}$$

The transition function δ' is defined as follows:

$$\begin{aligned} \delta' &= \{(q_1, \langle a, \gamma \rangle, q_2, \langle b, \gamma \rangle, \Delta) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma\} \\ &\cup \{(q_1, \langle \sigma', \gamma \rangle, q_2, \langle b, \gamma \rangle, \Delta) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\ &\cup \{(q'_1, \langle \gamma, a \rangle, q'_2, \langle \gamma, b \rangle, \bar{\Delta}) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma'\} \\ &\cup \{(q'_1, \langle \gamma, a \rangle, q_{\text{end}}, \langle \gamma, b \rangle, \bar{\Delta}) \mid (q, a, q_{\text{end}}, b, \Delta) \in \delta, q_{\text{end}} \in \{q_{\text{acc}}, q_{\text{rej}}\}, \gamma \in \Gamma'\} \\ &\cup \{(q'_1, \langle \gamma, \bar{\sigma}' \rangle, q'_2, \langle \gamma, b \rangle, \bar{\Delta}) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\ &\cup \{(q, >, q', >, R), (q', >, q, >, R) \mid q \in Q\}, \end{aligned}$$

where $\bar{L} = R, \bar{R} = L, \bar{>} = >$ and $\bar{<} = <$.