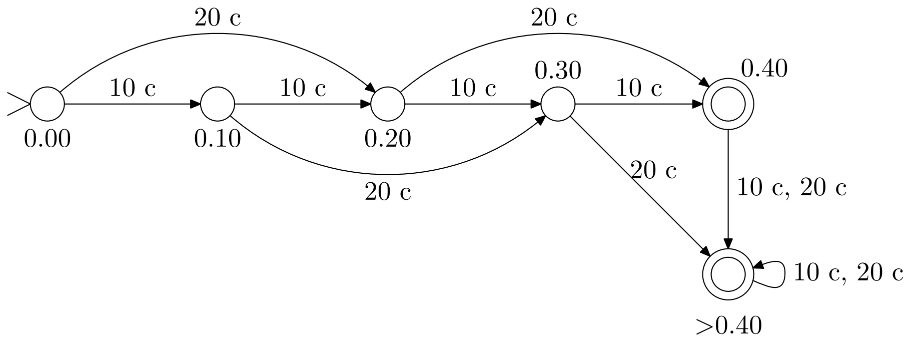
**T-79.1001/2** **Syksy 2005**
**Introduction to Theoretical Computer Science (T/Y)**
**Session 3**
**Answers to demonstration exercises**

4. **Problem**: Formulate the model of a simple coffee machine presented in class (lecture notes p. 15) precisely according to the mathematical definition of a finite automaton (Definition 2.1). What is the formal language recognized by this automaton?

**Solution**: The problem was to give formal representation to the finite automaton given below:



Formally, a deterministic state machine (finite automaton) is a tuple $M = (Q, \Sigma, \delta, q_0, F)$, where $Q$ is finite set of states, $\Sigma$ finite alphabet, $\delta$ function $Q \times \Sigma \to Q$, $q_0 \in K$ initial state and $F \subseteq Q$ set of accepting states.

In the Coffee Machine the parts are defined as follows:

$$\begin{aligned} Q &= \{0.00, 0.10, 0.20, 0.30, 0.40, {>}0.40\} \\ \Sigma &= \{10\text{c}, 20\text{c}\} \\ q_0 &= \{0.00\} \\ F &= \{0.40, {>}0.40\} \end{aligned}$$

The easiest way to show the transition function $\delta$ is a table.

| $q$ | $\delta(q, 10\text{c})$ | $\delta(q, 20\text{c})$ |
|---|---|---|
| 0.00 | 0.10 | 0.20 |
| 0.10 | 0.20 | 0.30 |
| 0.20 | 0.30 | 0.50 |
| 0.30 | 0.40 | >0.40 |
| 0.40 | >0.40 | >0.40 |

A configuration $c \in Q \times \Sigma^*$ of a finite automaton contains the current state and input. The automaton reads input one symbol at a time and then moves to a new state according to the symbol and current state. If the automaton is in an accepting state when the word ends, the word is accepted. Otherwise the word is rejected. The language $L(M)$ accepted by a finite automaton $M$ is the set of all accepted words. For the automaton in question it is:

$$L(M) = \{x_1 x_2 \ldots x_n \mid x_i \in \Sigma \text{ for all } 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} x_i \geq 40\ c\}$$

So the machine accepts all the strings in which the sum of given coins is 40 c or more. Let us see through a few inputs and how the machines operates with them.

- $w = 0.10\text{c}\ 0.10\text{c}\ 0.20\text{c}$:

$$\begin{aligned}(0.00, 0.10\text{c}\ 0.10\text{c}\ 0.20\text{c}) &\vdash_M (0.10, 0.10\text{c}\ 0.20\text{c}) \\ &\vdash_M (0.20, 0.20\text{c}) \vdash_M (0.40, \varepsilon)\end{aligned}$$

  Because $0.40 \in F$ the word is accepted. Here the $\vdash_M$ means that machine $M$ proceeds one step.

- $w = 0.20\text{c}\ 0.10\text{c}$:

$$(0.00, 0.20\text{c}\ 0.10\text{c}) \vdash_M (0.20, 0.10\text{c}\ ) \vdash_M (0.30, \varepsilon)$$

  Because $0.30 \notin F$, the word is rejected.

- $w = 0.20\text{c}\ 0.20\text{c}\ 0.20\text{c}$:

$$(0.00, 0.20\text{c}\ 0.20\text{c}\ 0.20\text{c}) \vdash_M^* (>0.40, \varepsilon)$$

  Word is accepted. Marking $\vdash_M^*$ denotes that machine $M$ proceeds zero or more steps.

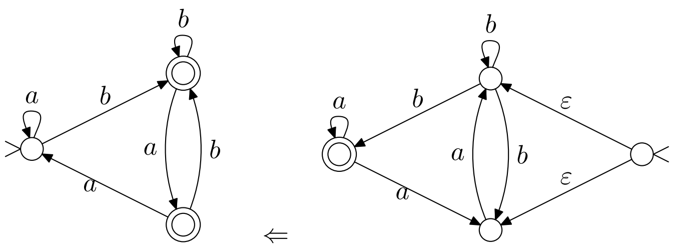
1. Show that if a language $L \subseteq \{a, b\}^*$ is recognized by some finite automaton, then so is the language $L^R = \{w^R \mid w \in L\}$. (The notation $w^R$ means the reverse of string $w$, cf. problem 2/2.)

2. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton that recognizes $L$ (that is, $L = L(M)$). We use it to form an automaton $M'$:

$$\begin{aligned}M' &= \{Q \cup \{q_0'\}, \Sigma, \delta', q_0', \{q_0\}\} \\ \delta' &= \{(q_i, a, q_j) \mid \delta(q_j, a) = q_i\} \\ &\qquad \cup \{(q_0', \varepsilon, q_i) \mid q_i \in F\},\end{aligned}$$

   where $q_0' \notin Q$.

   Intuitively this definition means that we get an automaton for $L^R$ by reversing all transitions of $M$ and by adding a new initial state that has empty transitions to all final states of $M$. The new automaton has only one final state, the original initial state.



6. **Problem**: Show that if languages $A$ and $B$ over the alphabet $\Sigma = \{a, b\}$ are recognized by some finite automata, then so are the languages $\bar{A} = \Sigma^* - A$, $A \cup B$, and $A \cap B$.

   **Solution**: Let $A, B \subseteq \Sigma^*$ be languages that can be recognized by finite automata. We now show that $\bar{A}$, $A \cup B$, and $A \cap B$ can also be recognized by finite automata by showing how such automata can be constructed.

   $\bar{A}$: Let $M_A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic automaton[1] that recognizes $A$ ($L(M_A) = A$). We define an automaton $M_{\bar{A}}$ as follows:

$$M_{\bar{A}} = (Q, \Sigma, \delta, q_0, Q - F) \ .$$

[1] $M_A$ necessarily exists since any nondeterministic finite automaton can be transformed into an equivalent deterministic one.

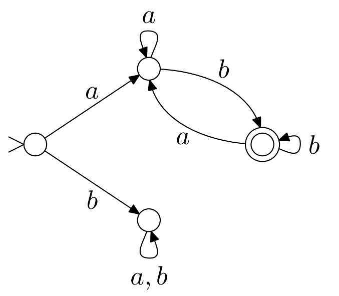
Automaton $M_{\bar{A}}$ works otherwise just as $M_A$, but its accecpting states are exchanged with its rejecting states. Thus, $M_{\bar{A}}$ accepts precisely those strings that $M_A$ rejects, and rejects those that $M_A$ accepts, so $L(M_{\bar{A}}) = \bar{A}$.

For example, consider an automaton that recognizes the language:

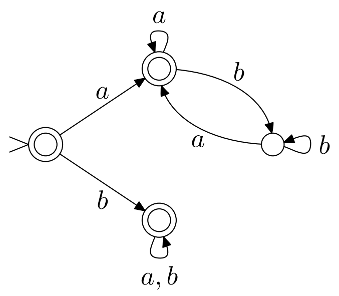
$$A = \{w \in \Sigma^* \mid \ w \text{ is of the form } axb, \text{ where } x \in \Sigma^*\} \ .$$

All strings that start with an $a$ and end with a $b$ are in $L$. The following two automata recognize languages $A$ and $\bar{A}$:

$M_A$:



$M_{\bar{A}}$:



Note that this construction works only if $M_A$ is deterministic. Try to find a counter example for the nondeterministic case.

$A \cup B$: Let $M_A = (Q_A, \Sigma, \delta_A, s_A, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, s_B, F_B)$ be finite automata that recognize languages $A$ and $B$. We suppose that the state sets are distinct, that is, $Q_A \cap Q_B = \emptyset$. This is not a serious limitation since the states of one of the automata can be renamed if necessary.

We construct a nondeterministic finite automaton $M_{A\cup B}$ as follows:

$$M_{A\cup B} = (Q, \Sigma, \delta, s, F) \ ,$$

where

$$\begin{aligned} Q &= Q_A \cup Q_B \cup \{s\} \\ F &= F_A \cup F_B \\ \delta &= \delta_A \cup \delta_B \cup \{(s, \varepsilon, s_a), (s, \varepsilon, s_b)\} \ . \end{aligned}$$

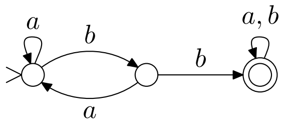
We construct $M_{A\cap B}$ by combining the automata $M_A$ and $M_B$. The state $s$ is a new initial state and from there is a nondeterministic $\varepsilon$-transition to initial states of $M_A$ and $M_B$.

If string $x \in A$, $M_{A\cup B}$ accepts it by first making a nondeterministic transition to $s_A$, and then doing the same sequence of transitions that $M_A$ would have done. Similarily, if $x \in B$, the first transition is to $s_B$.

For example, consider the automaton $M_A$ that was presented above and a new automaton $M_B$ that recognizes the language:

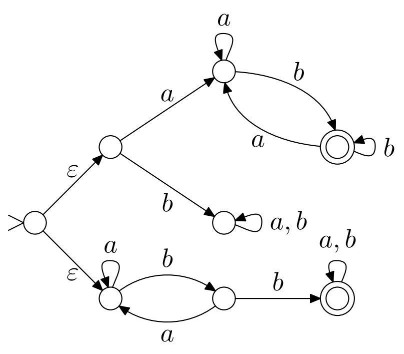
$$B = \{w \in \Sigma^* \mid w \text{ has a substring } bb\} \ .$$

$M_B$:



The language $A \cup B$ can be recognized by the following automaton:

$M_{A\cup B}$:



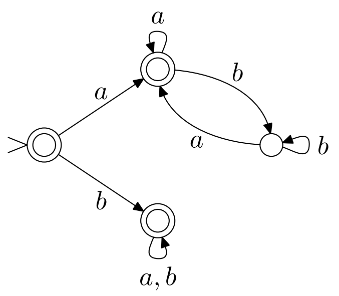
Often also a new final state $f$ and transitions $\{(f', \varepsilon, f) \mid f' \in F_A \cup F_B\}$ are added to $M_{A\cup B}$ . In this case $F = \{f\}$.

$A \cap B$: This claim is a corollary of the two previous constructions, since

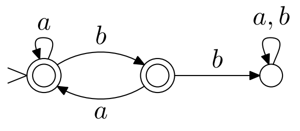
$$A \cap B = \overline{\overline{A} \cup \overline{B}} \ .$$

Let us examine again the above automata $M_A$ and $M_B$ construct $M_{A\cap B}$ using the above DeMorgan rule.

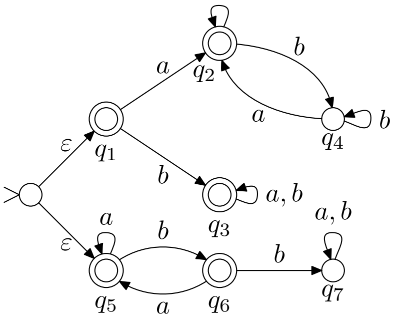
$M_{\bar{A}}$:



$M_{\bar{B}}$:

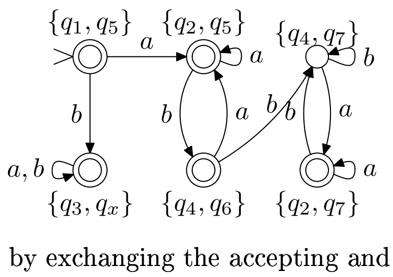


$M_{\bar{A}\cup\bar{B}}$:



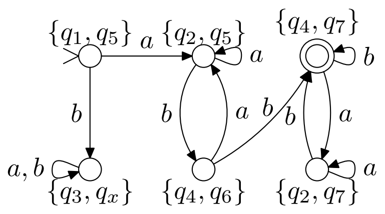
Before $M_{\bar{A}\cup\bar{B}}$ can be complemented, it has to be determinised (the following automaton is minimal, details of minimization are left in appendix):

$M'_{\bar{A}\cup\bar{B}}$:



We get the desired automaton by exchanging the accepting and rejecting states:

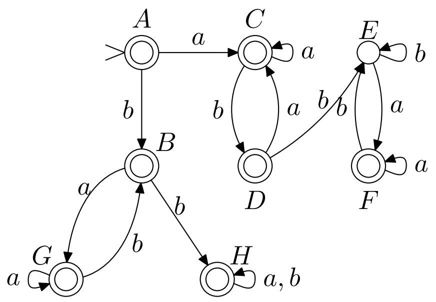
$M_{A\cap B}$:



We could also define the intersection of two automata directly, using a method that is analogous to the solution for the next exercise.

**Appendix: minimizing an automaton**

Using the determinising algorithm we can transform the automaton $M_{\bar{A}\cup\bar{A}}$ (of exercise 5) into the following form:



Now we want to find the minimal deterministic automaton that recognizes the same language. One algorithm is to define an equivalence relation $\overset{0}{\equiv}$ on the set of states and refine it step-by-step until we reach the desired relation $\equiv$.

In the first phase of the algorithm we remove all unreachable states. Since in this case all states are reachable, nothing has to be done.

Next, we construct the first equivalence partition such that all accepting states are in one class and all rejecting states in another:

0-equivalence:

| Class | State | $a$ | $b$ |
|---|---|---|---|
| I | $A$ | $C$ (I) | $B$ (I) |
| | $B$ | $G$ (I) | $H$ (I) |
| | $C$ | $C$ (I) | $D$ (I) |
| | $D$ | $C$ (I) | $E$ (II) |
| | $F$ | $F$ (I) | $E$ (II) |
| | $G$ | $G$ (I) | $B$ (I) |
| | $H$ | $H$ (I) | $H$ (I) |
| II | $E$ | $F$ (I) | $E$ (II) |

We see from the table that from the class I states $D$ and $F$ the $b$-transition leads to a state in class $II$, while for all other class I states the same transition leads to a class I state. So we separate the two distinct states into their own class:

1-equivalence:

| Class | State | $a$ | $b$ |
|---|---|---|---|
| I | $A$ | $C$ (I) | $B$ (I) |
| | $B$ | $G$ (I) | $H$ (I) |
| | $C$ | $C$ (I) | $D$ (III) |
| | $G$ | $G$ (I) | $B$ (I) |
| | $H$ | $H$ (I) | $H$ (I) |
| II | $E$ | $F$ (III) | $E$ (II) |
| III | $D$ | $C$ (I) | $E$ (II) |
| | $F$ | $F$ (III) | $E$ (II) |

This time states $C$ and $F$ do not fit in their classes and they have to be separated. This procedure is continued until all classes are consistent:

2-equivalence:

| Class | State | $a$ | $b$ |
|---|---|---|---|
| I | $A$ | $C$ (IV) | $B$ (I) |
| | $B$ | $G$ (I) | $H$ (I) |
| | $G$ | $G$ (I) | $B$ (I) |
| | $H$ | $H$ (I) | $H$ (I) |
| II | $E$ | $F$ (V) | $E$ (II) |
| III | $D$ | $C$ (IV) | $E$ (II) |
| IV | $C$ | $C$ (IV) | $D$ (III) |
| V | $F$ | $F$ (V) | $E$ (II) |

3-equivalence:

| Class | State | $a$ | $b$ |
|---|---|---|---|
| I | $A$ | $C$ (IV) | $B$ (VI) |
| II | $E$ | $F$ (V) | $E$ (II) |
| III | $D$ | $C$ (IV) | $E$ (II) |
| IV | $C$ | $C$ (IV) | $D$ (III) |
| V | $F$ | $F$ (V) | $E$ (II) |
| VI | $B$ | $G$ (VI) | $H$ (VI) |
| | $G$ | $G$ (VI) | $B$ (VI) |
| | $H$ | $H$ (VI) | $H$ (VI) |

All classes are now consistent so we can construct an automaton whose states are the equivalence classes. The minimized automaton is shown as a state diagram in the solution for exercise 5.

As a term, $k$-equivalence means that all states in a equivalence class treat all inputs that are at most $k$ symbols long in the same way; either they all accept the input or they all reject it.