T-79.1001/2 **Syksy 2005**
Introduction to Theoretical Computer Science (T/Y)
Session 2
Answers to demonstration exercises

4. **Problem**: Show that any alphabet Σ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over Σ can be easily encoded into strings over Γ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only *one* symbol, e.g. $\Gamma = \{1\}$?

   Solution: Let Σ be some alphabet with k symbols, $k > 1$. The strings of Σ can be coded as strings of $\Gamma = \{0, 1\}$ in the following manner.

   - Set the symbols of Σ to equal integers $\{1, \ldots, k\}$.
   - These numbers (the symbols of Σ) can be represented with binary numbers of length $\lceil \log_2 k \rceil$.
   - Every string in Σ^* can therefore be represented as a string of Γ by replacing the symbols of Σ with their binary encoding.

   The decoding from Γ^* to Σ^* can be done in a similar fashion by taking strings of length $\lceil \log_2 k \rceil$ from a string and interpreting them as symbols of Σ.

   If the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, the length of its counterpart $w' \in \Gamma^*$ is $|w'| = n \cdot \lceil \log_2 k \rceil$. This is because the number of symbols needed to encode any symbol in Σ is $\lceil \log_2 k \rceil$.

   For an example, consider the alphabet $\Sigma = \{a, b, c, d, e, f\}$ and the string $aacfd$. As $|\Sigma| = 6$, $\lceil \log_2 6 \rceil = \lceil 2.58 \rceil = 3$ bits are needed to represent the symbols of Σ. One possible encoding is

   $$\begin{array}{ll} a \mapsto 001 & d \mapsto 100 \\ b \mapsto 010 & e \mapsto 101 \\ c \mapsto 011 & f \mapsto 110 \end{array}$$

   With this encoding, the representation of $aacfd$ is 001001011110100.

   A similar coding scheme cannot be constructed if $\Gamma = \{1\}$. A unary presentation of the form $1 \mapsto 1$, $2 \mapsto 11$, $3 \mapsto 111$, ... can of course be defined, but the code obtained in this way can no longer be decoded unambiguously. For an example, the encodings of 1 1 1, 1 2, 2 1 and 3 are all the string 111.

5. **Problem**: Design finite automata that recognise the following languages:

   (a) $\{a^m b^n \mid m = n \mod 3\}$;

   (b) $\{w \in \{a, b\}^* \mid w \text{ contains equally many } a\text{'s and } b\text{'s, modulo } 3\}$.

   (The notation "$m = n \mod 3$" means that the numbers m and n yield the same remainder when divided by three.)

   Solution:

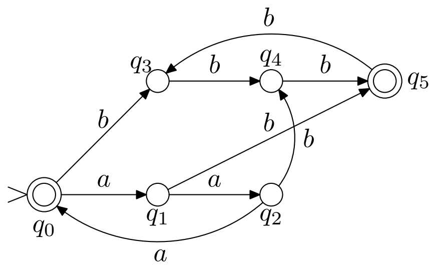
a) The language $L = \{a^m n^n \mid m = n \mod 3\}$ can be recognized by the finite automaton:

$$
\begin{aligned}
M &= (Q, \Sigma, \delta, q_0, F) \\
Q &= \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\} \\
\Sigma &= \{a, b\} \\
F &= \{q_0, q_5\}
\end{aligned}
$$

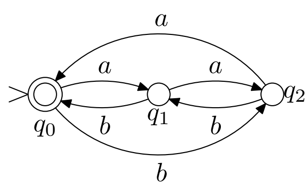
The state transition function δ is:

q	$\delta(q,a)$	$\delta(q,b)$
q_0	q_1	q_3
q_1	q_2	q_5
q_2	q_0	q_4
q_3	q_6	q_4
q_4	q_6	q_5
q_5	q_6	q_3
q_6	q_6	q_6

The state q_6 is used as a rejecting state. It means that the automaton moves into that state as soon as it is clear that the word cannot belong to the language (when ba substring is found). The machine then stays in that state until the end of the string. These states are often left out when an automaton is represented as a state diagram. This is also the case for the diagram below:

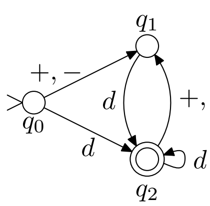


b) Language $L = \{w \in \{a,b\}^* \mid w$ contains as many a and b letters modulo $3\}$ is recognized by the following finite automaton:



6. **Problem**: Design a finite automaton that recognizes sequences of integers separated by plus and minus signs (e.g. $11+20-9$, $-5+8$). Implement your automaton as a computer program that also calculates the numerical value of the input expression.

Solution: The plus and minus operations can be recognized with the following automaton:



Here d is a shorthand notation that means any number from set $\{0, \ldots, 9\}$.

Below is a C-program that reads user input and does the required addition. It has been modified from the C-program presented in the lecture that can identify a number and it's size.

```
#include <stdio.h>
#include <ctype.h>

int main (void)
{
  int q;   /* state */
  int c;   /* input symbol */
  int sgn, val, sum;
  sgn=1; val = 0, sum = 0;
  q = 0;

  while ((c = getchar()) !='\n') {
    switch (q) {
    case 0:
      if (c == '+') q = 1;
      else if (c == '-') {
        sgn = -1;
        q = 1;
      }
      else if (isdigit(c)) {
        val = c -'0';
        q = 2;
      }
      else q = 99;
      break;

    case 1:
      if (isdigit(c)) {
        val = c -'0';
        q = 2;
      }
      else q = 99;
      break;

    case 2:
      if (isdigit(c)) {
        val = 10 * val + (c - '0');
        q = 2;
      }
      else if (c == '+')
        {
          sum = sum + val*sgn;
          val = 0;
          sgn = 1;
          q = 1;
        }
      else if (c == '-')
        {
          sum = sum + val*sgn;
```

```
          val = 0;
          sgn = -1;
          q = 1;
        }
      else q = 99;
      break;

    case 99:
      break;
    }
  }
  sum = sum + sgn*val;
  if (q == 2)
    printf("SUM IS %d.\n", sum);
  else
    printf("INVALID INPUT.\n");
  return;
}
```