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Note

On the coexistence of conference matrices and near resolvable 2-$(2k+1, k, k-1)$ designs

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Abstract

We show that a near resolvable 2-$(2k+1, k, k-1)$ design exists if and only if a conference matrix of order $2k+2$ does. A known result on conference matrices then allows us to conclude that a near resolvable 2-$(2k+1, k, k-1)$ design with even k can only exist if $2k+1$ is the sum of two squares. In particular, neither a near resolvable 2-(21, 10, 9) design nor does a near resolvable 2-(33, 16, 15) design exist. For $k \leqslant 14$, we also enumerate the near resolvable 2-$(2k+1, k, k-1)$ designs and the corresponding conference matrices.

Keywords: Conference matrix; near resolvable balanced incomplete block design; NRB; NRBIBD

1. Introduction

A *conference matrix* C of order n is a square $(0, \pm 1)$ matrix of side n with exactly one 0 in every row and every column such that $CC' = (n-1)I$, where I is the identity matrix and C' denotes the transpose of C. The only standard properties of conference matrices that we need are that transposing C or pre- or post-multiplying C by a signed permutation matrix

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still yields a conference matrix. For a fuller compilation of results on conference matrices, we refer the reader to Craigen's survey [2].

A t-(v, k, λ) design is a pair $(P, \mathcal{B})$, where P is a set of v *points* and $\mathcal{B}$ is a multiset of k-subsets of P—called *blocks*—such that every t-subset of P occurs in exactly λ blocks as a subset. A *parallel class* in a design is a set of blocks of the design that partitions the point set. A *near parallel class* is a set of pairwise disjoint blocks whose union has cardinality $v-1$. A (*near*) *resolution* is a partition of the multiset of blocks of a design into (near) parallel classes. A design is (*near*) *resolvable* if it has a (near) resolution. For more detailed results on near resolvable designs, we refer the reader to [4].

This paper establishes the following connection between near resolvable designs and conference matrices.

Theorem 1. *A near resolvable* 2-$(2k+1, k, k-1)$ *design exists if and only if a conference matrix of order* $2k+2$ *exists.*

By combining Theorems 1 with 2 below, we conclude the non-existence of near resolvable designs with certain parameter values.

Theorem 2 (*Craigen [2, §52.3.2]*). *For* $n \equiv 2 \pmod 4$, *a conference matrix of order* n *can exist only if* $n-1$ *is the sum of two squares.*

Geramita and Seberry state that Raghavarao [13] first proved Theorem 2, with another proof given later by van Lint and Seidel [9]. Following Ryser's suggestion, Geramita and Seberry [5, Theorem 2.10] gave a proof based on Lagrange's representation of $n-1$ by 4 squares, followed by reduction using Witt cancellation; so existence implies that there is a 2×2 rational matrix B such that $BB' = (n-1)I$, from which the sum of two squares criterion soon follows.

Corollary 3. *For even* k, *a near resolvable* 2-$(2k+1, k, k-1)$ *design can exist only if* $2k+1$ *is the sum of two squares.*

In particular, neither a near resolvable 2-(21, 10, 9) design nor a near resolvable 2-(33, 16, 15) design exists.

Theorem 2 only applies when $n \equiv 2 \pmod 4$, and the smallest non-prime power values of $n-1$ not excluded by Theorem 2 are $n-1 = 45$, 65 and 85. A conference matrix of order n does exist whenever $n-1 = 2k+1$ is an odd prime power—a near resolvable 2-$(2k+1, k, k-1)$ design is given by taking a block of the non-zero squares in $\mathrm{GF}(2k+1)$ and a block of the non-squares, and developing these two base blocks over $\mathrm{GF}(2k+1)$. Mathon [10] has constructed a conference matrix of order $n = 46$, so at least we know that $n-1 \equiv 1 \pmod 4$ does not have to be a prime power.

2. Properties of a near resolvable 2-$(2k+1, k, k-1)$ design

Every point in a 2-(v, k, λ) design must appear in λ blocks with every other point, so every point must appear in $r = \lambda(v-1)/(k-1)$ blocks, and the total number of blocks must be $b = vr/k$. So, in a 2-$(2k+1, k, k-1)$ design we have $r = 2k$ and $b = 4k+2$. Next we consider a near resolution of the design. We take some block and look at the number of points that it has in common with the other blocks outside its near parallel class—we know that it has no points in common with its near parallel classmate. Suppose it has i points in common with n_i other blocks outside its near parallel class. Counting blocks, incidences and pairs, we have

$$\sum_i n_i = b - 2, \qquad \sum_i i n_i = k(r-1), \qquad \sum_i \binom{i}{2} n_i = (\lambda - 1)\binom{k}{2}. \tag{1}$$

Substituting for b, r and λ and rearranging give us

$$\sum_{i=0}^{k} n_i (i-\alpha)(i-\alpha-1)/2 = 2k\,(\alpha - (k-1)/2)\,(\alpha - (k-2)/2)\,. \tag{2}$$

Lemma 4. *In any near resolvable* 2-$(2k+1, k, k-1)$ *design, the pattern of intersection sizes is the same for every block. The pattern is*

$$0^1 (t-1)^k t^{3k} \quad \textit{for even } k = 2t; \textit{ and} \tag{3}$$
$$0^1 (t-1)^{3k} t^k \quad \textit{for odd } k = 2t-1. \tag{4}$$

Proof. Substitute $\alpha = t-1$ into (2). Since α is an integer, every summand on the left-hand side of (2) is non-negative, and as the right-hand side is zero, it follows that $n_i = 0$ whenever $i \notin \{t-1, t\}$. The result follows from solving n_{t-1} and n_t from (1). □

Remark 5. Since a near resolvable 2-$(2k+1, k, k-1)$ design has $b = 4k+2$ blocks, and every near parallel class contains two blocks, there are $b/2 = 2k+1$ near parallel classes. As each point appears in $r = 2k$ near parallel classes, every point must miss (i.e., be the hole for) exactly one near parallel class.

3. Proof of Theorem 1

A square $(0, \pm 1)$ matrix M with side $2k+1$ is a *matrix representation* of a 2-$(2k+1, k, k-1)$ near resolution if it can be obtained as follows. Label the points of the design arbitrarily as $i \in \{1, 2, \ldots, 2k+1\}$. Label the near parallel classes arbitrarily as $j \in \{1, 2, \ldots, 2k+1\}$. Within each near parallel class j, label the blocks arbitrarily as B_j^- and B_j^+. Now, let $m_{ij} = -1$ if $i \in B_j^-$, $m_{ij} = 1$ if $i \in B_j^+$, and $m_{ij} = 0$ otherwise.

The general structure of the proof is as follows. In Lemma 6 it is shown that every matrix that satisfies certain conditions is a matrix representation of a 2-$(2k+1, k, k-1)$ near resolution. In Lemma 7, it is shown that by removing a row and a column from a conference matrix of order $2k+2$ in a certain manner one can obtain a matrix that satisfies these conditions. In Lemma 8, it is shown that every matrix representation of a 2-$(2k+1, k, k-1)$ near resolution can be augmented to a conference matrix of order $2k+2$. Theorem 1 then immediately follows from Lemmas 7 and 8.

Lemma 6 is an application of a theorem of Haanpää and Kaski [7], who consider an analogous representation for near resolvable designs with $v = qk + 1$. We denote by J the matrix of the appropriate size with all entries equal to one.

Lemma 6. *If $MM' = (2k+1)I - J$ for a square $(0, \pm 1)$ matrix M of side $(2k+1)$, then M is the matrix representation of a 2-$(2k+1, k, k-1)$ near resolution.*

Proof. Because $MM' = (2k+1)I - J$, it follows that M has exactly one zero in every row and, since the inner product of any pair of rows has an odd number of non-zero summands, there is exactly one zero in every column. We know that M represents some near resolvable design. We know that this design has the correct replication number and block count for a near resolvable 2-$(2k+1, k, k-1)$ design. It remains to show that the block size is constant and equal to k, and that every pair of points occurs in $k-1$ blocks.

Pre-multiplying $MM' = (2k+1)I - J$ by $1'$ and post-multiplying by 1, we find that $(1'M)(1'M)' = 0$, so the vector $(1'M) = 0'$. So every column must contain one value 0, k values of $+1$ and k values of -1, and thus the blocks represented by M all have size k.

Next, consider the inner product of any two distinct rows, R_i and R_j, of M. Let λ_{ij} be the number of times R_i and R_j have the same element in a column. Then

$$R_i R_j' = 2 \cdot 0 + (2k + 1 - \lambda_{ij} - 2)(-1) + \lambda_{ij}(+1) = 2\lambda_{ij} - (2k - 1). \tag{5}$$

By hypothesis, $R_i R_j' = -1$, thus $\lambda_{ij} = k - 1$. In other words, every pair of distinct points, i and j, appear in the same block $k-1$ times in the design given by M. $\square$

Lemma 7. *If there exists a $2k+2$ conference matrix, then there exists a near resolvable 2-$(2k+1, k, k-1)$ design.*

Proof. Choose a zero in the conference matrix. We can pre- and post-multiply our conference matrix by signed permutation matrices so that our chosen zero is mapped to the top left corner of the resulting conference matrix C, which we may write in the following form:

$$C = \left(\begin{array}{c|c} 0 & x' \\ \hline 1 & M \end{array}\right). \tag{6}$$

Since

$$CC' = \left(\begin{array}{c|c} x'x & x'M' \\ \hline Mx & J + MM' \end{array}\right) \tag{7}$$

and $CC' = (2k+1)I$, we have $MM' = (2k+1)I - J$, and so M satisfies the condition of Lemma 6, and hence M is the matrix representation of a 2-$(2k+1, k, k-1)$ near resolution. $\square$

Lemma 8. *If there exists a near resolvable 2-$(2k+1, k, k-1)$ design, then there exists a conference matrix C of order $2k+2$.*

Proof. Let M be a $(0, \pm 1)$ matrix representation of a near resolution of the design. First, we note that every column of M contains exactly k values of $+1$, k values of -1 and one value of 0, so $JM = 0$. Also, the inner product of any two distinct rows of M, say R_i and R_j, satisfies (5) and, since $\lambda_{ij} = k - 1$, we have $R_i R_j' = -1$. Thus $MM' = (2k+1)I - J$.

Form C by augmenting M as in (6). Now, if we pick x as a (± 1) vector, then C will be a $(0, \pm 1)$ matrix with exactly one zero in every row and column, and if we also have $Mx = 0$, then (7) shows that $CC' = (2k+1)I$ also holds and C will be a conference matrix.

We next show that $X = (2k+1)I - M'M$ is a (± 1) matrix. Consider two near parallel classes, consisting of the blocks A and B, and C and D, and suppose A and C are the blocks that contain the holes of the other near parallel class. We will consider the frequencies of the possible row pairings using Lemma 4.

Column i	0	A	A	A	B	B
Column j	C	0	C	D	C	D
Frequency for $k = 2t$	1	1	$t-1$	t	t	t
Frequency for $k = 2t-1$	1	1	$t-1$	$t-1$	$t-1$	t

Now the value of the inner product depends on the parity of k and whether the signs of A and C differ or not, but the only possible values for the inner product of distinct columns of M are $+1$ and -1. Thus X is a (± 1) matrix with $+1$'s on the diagonal. Also, $MX = (2k+1)M - MM'M = (2k+1)M - ((2k+1)I - J)M = JM = 0$. So we may pick x as any column of X to form our conference matrix. □

There is an alternative proof of the fact that $X = (2k+1)I - M'M$ is a (± 1) matrix which might be of interest. We rely on the fact that X is symmetric with unit diagonal, and $MX = 0$. We find $XX = (2k+1)X$ by post-multiplying the defining equation for X by X. Now X is symmetric and so has real eigenvalues, and post-multiplying $XX = (2k+1)X$ by an eigenvector shows that the eigenvalues are 0 and $2k+1$ with frequencies $2k$ and 1 (the frequencies follow by considering the trace of X). So X has rank 1, and thus X has the form yy' (recalling the symmetry) and $x_{ii} = y_i^2$ (the alternative that $X = -yy'$ would imply $x_{ii} = -y_i^2$). So every element of y is ± 1, and by picking x as a column of X, we are either picking $x = y$ or $x = -y$. In either case, x is a (± 1) vector and $X = xx'$.

4. Analogies with the Hadamard case

Our aim in this section is to correlate our work in the previous sections with some fairly well-known results in the analogous Hadamard case. By pursuing this analogy, we are able to obtain Theorem 9 below, an apparently new result.

An *Hadamard matrix* H of order n is a square (± 1) matrix of side n such that $HH' = nI$. If we apply a similar approach for resolvable 2-$(2k, k, k-1)$ designs, with a similar representation by M, with the H (of order $2k$) partitioned as $H = (1 \mid M)$, then we obtain some familiar results. The design has $b = 4k - 2$ blocks and $r = 2k - 1$. The block intersection pattern for every block in the resolvable design is $0^1(k/2)^{2k-2}$; k must be even if $k > 1$, and no resolvable 2-$(2k, k, k-1)$ design exists if $k > 1$ is odd—almost correspondingly, the order of an Hadamard matrix must be 1, 2 or a multiple of 4. If we sign the columns of H (equivalently, M) so that the first row is all $+1$, then, by row orthogonality, we obtain more information than that any pair of rows (excluding the first row) of H have

k identically signed columns: we actually have each of the 4 possible plus/minus patterns occurring $k/2$ times. This is essentially the usual proof that if H has three or more rows, then its order must be a multiple of 4. Now this extra information allows us to conclude that if we remove the first row and column, then the pattern of $+1$'s, $(J+H)/2$, gives the incidence matrix of a 2-$(n-1, n/2-1, n/4-1)$ design, and the pattern of -1's, $(J-H)/2$, gives the incidence matrix of a 2-$(n-1, n/2, n/4)$ design. Now the missing row 1 point from these two designs occurs with every pair $n/4-1$ times from the $+1$ incidences, and the signing that made row 1 special merely amounted to labelling the block containing row 1 as the first (i.e., $+1$ block) in every parallel class, and clearly we could relabel the blocks within each parallel class arbitrarily, and so the property that a point occurs with every pair $n/4-1$ times holds for all points and M actually represents a resolvable 3-$(2k, k, k/2-1)$ design, and so a resolvable 3-$(2k, k, k/2-1)$ design and an Hadamard matrix of order $2k$ co-exist.

Next, suppose we take a conference matrix C of order n and permute the rows so that the zeros all fall on the diagonal, and then resign the rows and columns so that the off-diagonal elements in the first column are all $+1$, and the off-diagonal elements in the first row are constant. If we try to repeat the above argument for conference matrices, we have some difficulty as we do not know what is in the rest of the column when we are considering the zero in some row—the row orthogonality gives us no information here. However, we can conclude that n must be even if $n>1$, and that M must be symmetric if $n \equiv 2 \pmod 4$ and $n>2$, and skew-symmetric (i.e., $M' = -M$) if $n \equiv 0 \pmod 4$, and we can extend this to the whole matrix C by the choice of sign for the first row, a result due to Delsarte et al. [3].

We can extend the Hadamard 3-design result to a corresponding result on conference matrices, also with the best possible index.

Theorem 9. *If a near resolvable* 2-$(2k+1, k, k-1)$ *design exists, then a resolvable* 2-$(2k+2, k+1, 2k)$ *design exists which is a* 3-$(2k+2, k+1, k-1)$ *design. Moreover, if k is odd, then these indices may be halved.*

Proof. The construction of the 2-design is to take one copy of the design and to adjoin a new point, say $\{\infty\}$, to every block. Clearly every finite pair will occur $k-1$ times, and the infinite point will appear with every finite point $2k$ times and, in fact, will appear with every finite pair $k-1$ times. For the second half of the design, we can adjoin the hole to each block in its near resolution class which gives a 2-$(2k+1, k+1, k+1)$ design on the finite points (see [4, Lemma 4.2.17]).

If the underlying conference matrix for the near resolvable design is written as in (6), with M having a zero diagonal and $x=1$, then the design on the finite points given by our construction can be written in compressed form as $(M+I|M-I)$. The infinite point will add an extra row $(-1'|1')$ to this matrix. Now, if the conference matrix of order $2k+2$ has a zero diagonal with its off-diagonal elements in the first row and column constant, then M will necessarily be symmetric if k is even, and skew if k is odd. If k is odd, then adding a column of ones to each part of the augmented partition produces a pair of Hadamard matrices, and our result follows directly in this case. If k is even, using the symmetry one can compute the number of columns of $(M+I|M-I)$ which have a triple of same signed elements. This computation is quite straightforward (but messy, so we will omit the details) and establishes that any triple of finite points occurs $k-1$ times, and we have shown above that the triples involving an infinite point also occur $k-1$ times. □

Remark 10. We have not been able to determine whether conference matrices have been used to construct 3-designs as in Theorem 9. The result for Hadamard matrices is well-known (see e.g., [8, Remark 3.20]). If k is even and $2k+1$ is a prime power, then the 3-design is known [8, Table 3.31], but our result for $k = 22$, for example, appears to be new.

5. Enumeration up to isomorphism

In this section, we enumerate the near resolvable 2-$(2k+1, k, k-1)$ designs up to isomorphism for $k \leqslant 14$ and the corresponding conference matrices. We take advantage of the connection between the designs and conference matrices.

Recall that two conference matrices are considered equivalent if one can be mapped to the other by pre- and post-multiplying by a signed permutation matrix. The following theorem is from Delsarte et al. [3].

Theorem 11. *Every conference matrix C of order $2k+2$ is equivalent to a symmetric ($C = C'$) or a skew ($C = -C'$) one according to whether k is even or odd.*

Two matrix representations of 2-$(2k+1, k, k-1)$ near resolutions are equivalent if one can be obtained from the other by pre-multiplying with an unsigned permutation matrix—corresponding to a permutation of the points in the design—and by post-multiplying with a signed permutation matrix—corresponding to the arbitrary labelling of near parallel classes in the near resolution and the blocks within them. Unlike conference matrices, here negation of individual rows is not allowed. Nevertheless, an analogue of Theorem 11 holds for matrix representations of 2-$(2k+1, k, k-1)$ near resolutions:

Lemma 12. *Every matrix representation of a* 2-$(2k+1, k, k-1)$ *near resolution is equivalent to a symmetric or a skew one according to whether k is even or odd.*

Proof. Let M be a matrix representation of a 2-$(2k+1, k, k-1)$ near resolution. We may extend M to a conference matrix C as in the proof of Lemma 8. By Theorem 11, C is equivalent to a symmetric or skew conference matrix under pre- and post-multiplication by a signed permutation matrix. For any signed permutation matrix S, SC is symmetric (respectively, skew) if and only if CS is, so it suffices here to consider post-multiplication only, and CS is symmetric or skew for some signed permutation matrix S. Now S must preserve the location of the first column, and by removing the first row and column from S we obtain another signed permutation matrix T such that MT is symmetric or skew as appropriate. □

We use an orderly algorithm analogous to [7] to enumerate the near resolvable 2-$(2k+1, k, k-1)$ designs. We construct square $(0, \pm 1)$ matrices of side $2k+1$ row by row, and require that the matrices be symmetric or skew according to the parity of k, that the number of $+1$'s and -1's in every row be exactly k, and that the inner product of any two distinct rows be -1. As a technical detail, to guarantee that the lexicographically minimum partial matrix being constructed has zeroes on the diagonal we order the possible matrix rows primarily by the location of the zero and secondarily by their usual lexicographical order. The isomorph rejection in our backtrack search only considers matching permutations of rows and columns; these form a subgroup of the group formed by the actual equivalence operations of matrix representations, since negating columns is also allowed. Consequently, the backtrack search may generate isomorphic matrix representations. We therefore encode the designs obtained by our search as graphs and use nauty [11] for isomorph elimination.

For $k \leqslant 8$ the near resolvable 2-$(2k+1, k, k-1)$ design is unique up to isomorphism, and for $k = 9, 10, \ldots, 14$ there are 2, 0, 19, 8, 374, 21 pairwise nonisomorphic designs, respectively. Our results agree with the results reported by Morales et al. [12] for $k \leqslant 13$.

As a consistency check, we can use Lemmas 7 and 8 to enumerate the conference matrices based on our designs and again obtain the designs from the conference matrices. For $k \leqslant 8$ the conference matrix of order $2k+2$ is unique, while for $k = 9, 10, \ldots, 14$ there are 2, 0, 9, 4, 41, 6 pairwise inequivalent conference matrices, respectively. As a further consistency check, each matrix M constructed by our backtrack search can be interpreted as the adjacency matrix A of a particular class of graphs by letting $A = (M + J - I)/2$. For symmetric M, A is the adjacency matrix of a class of strongly regular graphs known as conference graphs, and for skew M, A is the adjacency matrix of a doubly regular tournament. For $k = 2, 4, \ldots, 14$ there are, respectively, 1, 1, 1, 1, 0, 15, 41 pairwise non-isomorphic conference graphs on $2k+1$ vertices, which agrees with the results quoted by Brouwer [1]. For $k = 1, 3, \ldots, 13$ there are, respectively, 1, 1, 1, 2, 2, 37, 722 pairwise non-isomorphic doubly regular tournaments on $2k+1$ vertices. This agrees with the enumeration of certain equivalent structures based on skew Hadamard matrices (Spence's $D(n)$) by Spence [14].

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