Helsinki University of Technology Laboratory for Theoretical Computer Science
Research Reports 62
Teknillisen korkeakoulun tietojenkäsittelyteorian laboratorion tutkimusraportti 62
Espoo 2000 HUT-TCS-A62

COVERING A SQUARE WITH UP TO 30 EQUAL CIRCLES

Kari J. Nurmela and Patric R. J. Östergård

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Laboratory for Theoretical Computer Science
Research Reports 62
Teknillisen korkeakoulun tietojenkäsittelyteorian laboratorion tutkimusraportti 62
Espoo 2000 HUT-TCS-A62

COVERING A SQUARE WITH UP TO 30 EQUAL CIRCLES

Kari J. Nurmela and Patric R. J. Östergård

Helsinki University of Technology
Department of Computer Science and Engineering
Laboratory for Theoretical Computer Science

Teknillinen korkeakoulu
Tietotekniikan osasto
Tietojenkäsittelyteorian laboratorio

Distribution:
Helsinki University of Technology
Laboratory for Theoretical Computer Science
P.O.Box 5400
FIN-02015 HUT
Tel. +358-0-451 1
Fax. +358-0-451 3369
E-mail: lab@tcs.hut.fi

ISBN 951-22-5072-1
ISSN 0783-5396

Picaset Oy
Helsinki 2000

Abstract: A computational method for finding good coverings of a square with equal circles is presented. The algorithm uses a quasi-Newton method with BFGS secant update to minimize the uncovered area by moving the circles. The radius of the circles is further adapted to find locally optimal coverings, and the algorithm is applied repeatedly to random initial configurations. The structures of the coverings are determined and the coordinates of each circle are calculated with high precision using a mathematical model for an idealized physical structure consisting of tensioned bars and frictionless pin joints. Best found coverings of a square with up to 30 circles are displayed, 17 of which are new or improve on earlier published coverings.

Keywords: circle covering, equidistant graph, square

CONTENTS

1 INTRODUCTION

Packing circles in various shapes in the plane or on a sphere has been a popular research subject of discrete geometry during the last few decades. In such packing problems one tries to find a configuration of n equal non-overlapping circles inside the area so that the radius of the circles is maximized. Good packings have been found, for example, in a square, in an equilateral triangle, in a circle, and on a sphere (see [2, 9] for references). On the other hand, the dual problem of determining good *coverings* of an area with n equal circles has received much less attention.

Most results in the literature on coverings of geometrical shapes concern the problem of covering a sphere by circles (circular caps); see [5, 9] and their references. Recently, however, there has been a growing interest in studying good coverings of several other shapes, mainly in the plane. The earliest computer results for circle coverings in the plane were obtained by Zahn, Jr. [19], who searched for good coverings of a circle by equal circles. Zahn, Jr. made the problem discrete by dividing the big circle into small cells and then applying local optimization methods to cover as many of these cells as possible while keeping the radius of the small circles constant.

The first computational results on coverings of a square appeared in 1995 when Tarnai and Gáspár published coverings with up to 10 circles [17]. They used an engineering approach where a locally optimal covering is found by simulating a system consisting of shrinking, tensioned bars and pin joints. Melissen and Schuur [10] improved the coverings with 6 and 8 circles and presented a new covering with 11 circles. Their approach is based on a simulated annealing algorithm using the Voronoi tessellation of the square with respect to the centers of the circles. Finally, extending the work in [17], a study by Lengyel and Veres [7] gives coverings of a square with up to 23 circles.

Other related results include coverings of a rectangle with up to 7 circles [6, 11], coverings of an equilateral triangle (with optimality proofs in the smallest cases) [8, 12], and coverings of a circle [13]. A survey on circle coverings can be found in [9, Ch. 5].

In Section 2 we present a new algorithm for finding good circle coverings of a square. A method for numerically determining the structure of a covering is presented in Section 3. Using these algorithms, in Section 4 we improve the previous coverings of a square with 12 to 21 circles [7] and present new coverings with 24 to 30 circles. All best known coverings with up to 30 circles are depicted.

2 THE COVERING ALGORITHM

In this work we present an algorithm that can be used to find locally optimal coverings of a square with n equal circles. The algorithm consists of two nested levels: on the inner level, the radius of the circles is constant and an optimization routine is used to minimize the uncovered area of the square by moving the circles. The outer level of the algorithm adjusts the radius of the circles depending on whether or not a covering was found on the inner level.

The algorithm can be generalized to cover any area in the plane.

The core of the algorithm is a local optimization routine that tries to cover the square starting from a given configuration of circles (the radius of the circles is constant during this optimization phase). The function to be minimized is the uncovered area of the square (a discrete version of this function was used in [19]). We have implemented routines that calculate the uncovered area and the first partial derivatives with respect to the coordinates of the centers of the circles. This allows us to use as a local optimizer the subroutine SUMSL of [3], an implementation of a quasi-Newton method using the well-known BFGS secant update to approximate the second derivatives. In this work this local optimization routine is called "the BFGS routine". The nature of the objective function requires a reliable line search; in [3] a double dogleg technique with trust region is used.

With this local optimization procedure we start from a random initial circle configuration and try to cover the square. If a covering is found, the radius of the circles is decreased and the optimization is continued. If the local optimizer converges to a partial covering, we increase the radius and restart the optimization from the last covering found.

Control of the radius r is not trivial, because we have to make compromises in the convergence criteria: the BFGS routine may stop with a partial covering even though a covering would be obtainable with some extra iterations. That is, failure in finding a covering does not imply that a covering with equal or smaller radius cannot be found later, after additional BFGS runs and adjustments of r. The algorithm can be outlined as follows:

1. Choose an initial value for the circle radius r and start a BFGS optimization run from a random initial configuration. Store the resulting configuration in S. Initially, let $\alpha := 0.9$.

2. If S is not a covering, go back to step 1. Otherwise, let $r' := \alpha r$.

3. Start a BFGS optimization run from the configuration S with radius r'. Store the resulting configuration in S'.

4. Check if S' is a covering with radius r'.

   (a) Covering was achieved: Let $S := S'$ and $r := r'$. Adjust α and let $r' := \alpha r'$. Go to 3.

   (b) Covering was not achieved: If $r - r' < \delta$, stop the run. Otherwise, let $r' := (r + r')/2$. Go to 3.

In the first step we take a random initial configuration of circles with radius larger than in the best currently known covering. We then move the circles using the BFGS routine and hopefully end up with a covering. If a covering is not found, it seems probable that we are near a local optimum with a greater radius than the initial value of r. In this case we restart the algorithm with a different initial configuration. Otherwise, we store the covering in S and the corresponding radius in r. The new candidate radius is r'. Throughout the rest of the run, the best covering (smallest radius) found so far is stored in S and the corresponding radius in r. We always have $r' \leq r$.

If a covering with radius r' is found in step 3, the value of α is adjusted in the following way. If the previous BFGS run (with radius r'') also found a covering, then we let

$$\alpha := \max(\alpha^2, 0.9).$$

If the previous BFGS run with radius r'' did not find a covering (in this case $r'' < r'$), we let

$$\alpha := \frac{r' + r''}{2r'}.$$

If, at any stage of the algorithm, a BFGS run converges to only a partial covering when the radius r is larger than in the best currently known covering, we can stop the run (we are near a local optimum different from the global optimum). This greatly speeds up the search procedure, because we can often very fast reject initial solutions leading to a poor local optimum. The better upper bound we have for the optimal radius, the better we can apply this technique. Each record-breaking covering improves the upper bound.

Another way to speed up the algorithm is to limit the number of iterations in the BFGS routine in the early stages of the algorithm. Namely when r is not (yet) very accurate, it is unnecessary to calculate the local optimum in the BFGS routine with high precision.

We apply the algorithm repeatedly starting from random initial configurations and selecting the best covering found. This stochastic global optimization method is called a *multistart* algorithm. When the number of variables is not too large it usually gives good results, provided that enough optimization runs with different initial configurations are performed.

Finding a good stopping criterion is a very hard problem in stochastic global optimization algorithms. We used the following approach: for $n \leq 27$ we stopped the search when the best covering had been found 80 times. The largest number of runs were performed for $n = 27$, where 160000 runs (taking about two weeks of CPU time on a 233 MHz Pentium PC running Linux operating system) from random initial solutions were required to find the solution presented in this work 80 times. For most of the smaller values of n, a few hours of CPU time was sufficient.

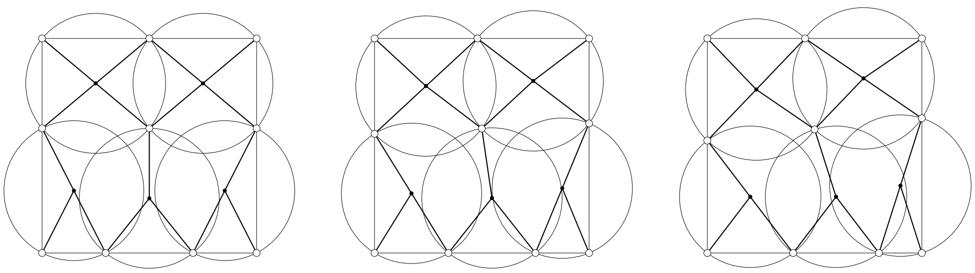
For $28 \leq n \leq 30$ it seems that very large amounts of CPU time would be needed to find the solutions as many as 80 times when starting from random initial solutions (for $n = 28$ this would probably require over two million runs). Only 100000 runs were performed for each $28 \leq n \leq 30$; the best coverings with 28, 29, and 30 circles were found 3, 21, and 15 times, respectively. No searches were performed for $n > 30$ circles, although with larger computing resources the algorithm can be used also for larger n.

A successful optimization run that finds a covering not exceeding the prescribed value of r takes at most a few minutes of CPU time when $\delta = 10^{-7}$ in our algorithm. If the initial solution is poor, so that the optimization algorithm is likely to converge to a covering inferior to the best known covering, then the run can in most cases be ended in a couple of seconds.

3 OBTAINING NUMERICAL SOLUTIONS

The covering algorithm of the previous section can provide the coordinates of a covering only with an accuracy limited by the precision of the computations. It is important to compute the positions of the circles with high precision to validate the assumed structure of the covering.

To find the coordinates of a locally optimal covering with greater accuracy, we identify the points where at least three circles (or two circles and the square boundary, or one circle and a corner of the square) appear to intersect. By inserting edges between these points and the centers of the circles, we get the *graph* of the covering. By requiring that each edge of the graph has length equal to r we get a system of nonlinear equations, which has a solution at a locally optimal covering. However, this system of equations is usually underdetermined and the value of r cannot be determined from the underlying graph alone. There is a continuous movement of the circles such that each covering along the path of the movement satisfies the system of equations; an example of this is given in Figure 1. In some cases ($n = 7$, 10, 22, 27) the system of equations obtained in this way has more equations than variables but still does not have a unique solution in a finite neighborhood of the local optimum—there are redundant equations in the system.



$r = 0.32616058\ldots$ $r = 0.32690747\ldots$ $r = 0.32956817\ldots$

Figure 1: Three different coverings having the same underlying graph.

In [9, 10] additional equations are obtained by differentiating the original equations with respect to one or more variables of the original system (this also increases the number of variables). In this work we use a more easily automated method: we use a mathematical model for an idealized physical structure with tensioned bars and frictionless pin joints, cf. [17]. Each edge in the graph of the covering corresponds to a bar and each vertex corresponds to a pin joint. Now if each bar is tensioned, the system starts to move towards a locally optimal covering, provided that the tensions are adjusted so that all bars have equal length. In a configuration corresponding to a locally optimal covering, all pin joints (and the square) are in rest while most (but not necessarily all) bars have tension greater than zero. If a bar is in compression instead of tension, then the graph does not correspond to a locally minimal covering [17].

We write down the equations of motion for each pin joint and the square and add these equations to the system of equations formed earlier. We now also have new variables corresponding to the tensions in the bars. The total tension of the structure is not determined, so we arbitrarily set the tension of a suitably selected bar equal to a constant. At no stage we impose symmetry constraints on the solution of the system; any symmetries are investigated only after the system is solved numerically.

The largest system of equations in this work has 286 variables (the covering with 27 circles). In comparison, solving numerically the coordinates of the best known packing of 27 circles in a square involves only 55 variables [15]. This may suggest that a covering problem with n circles is in some sense more difficult than the corresponding packing problem with the same number of circles.

The optimization algorithm in Section 2 provides us with very good initial values for the coordinates of the pin joints (the intersections and the centers of the circles), but the tensions in the bars are not known. However, this difficulty is easily bypassed by solving the initial tensions in a separate step before solving the whole system. By taking the equations of motion, we can form a linear system of equations from which the tensions can be solved. Usually the system is overdetermined and we only have approximate locations of the pin joints, so the system cannot be solved exactly; we use a least squares solution.

After solving the initial tensions we have a good initial solution for the whole system of equations. In the numerical solution we use a modified Newton-Raphson method [1], which works even when the system is overdetermined. All systems in this work were solved with such an accuracy that the maximal error in the equations was less than 10^{-100} (in the computations the side of the square was equal to 2 and one of the tensions was fixed to 1), using a mathematical software supporting arbitrary precision computations [18].

In the modified Newton-Raphson method, the most time-consuming step of each major iteration is the calculation of the pseudoinverse of the system. However, it is sufficient to calculate the pseudoinverse only with machine precision, which speeds up the procedure considerably and still allows us to gain 10 to 14 digits of precision in the solution in each major iteration of the modified Newton-Raphson method.

To determine whether the solution found is unique in a finite neighborhood of the initial solution we solve the system several times with slightly perturbed initial solutions, cf. [4]. Since in all coverings in this work the solution converged each time to the same solution within the high precision used, we conclude that (with high probability) the coverings with presented structure exist and they are true (at least locally) optimal coverings. We now also have very accurate numerical values for the corresponding values of r.

4 THE RESULTS

The best coverings found are depicted in Figures 2–6. For completeness, we show configurations for all $1 \leq n \leq 30$. The coverings with $n \leq 5$, $n = 7$ are known to be optimal [6]. The coverings for $n = 1, 2, 3, 4, 5, 7, 9$, and 10 have earlier been published in [17], and the coverings for $n = 6, 8$, and 11 in [10]. The covering for $n = 11$ was also given in [7], as well as those for $n = 22$ and 23 (the latter two were found independently also in this study). All the other coverings are new.

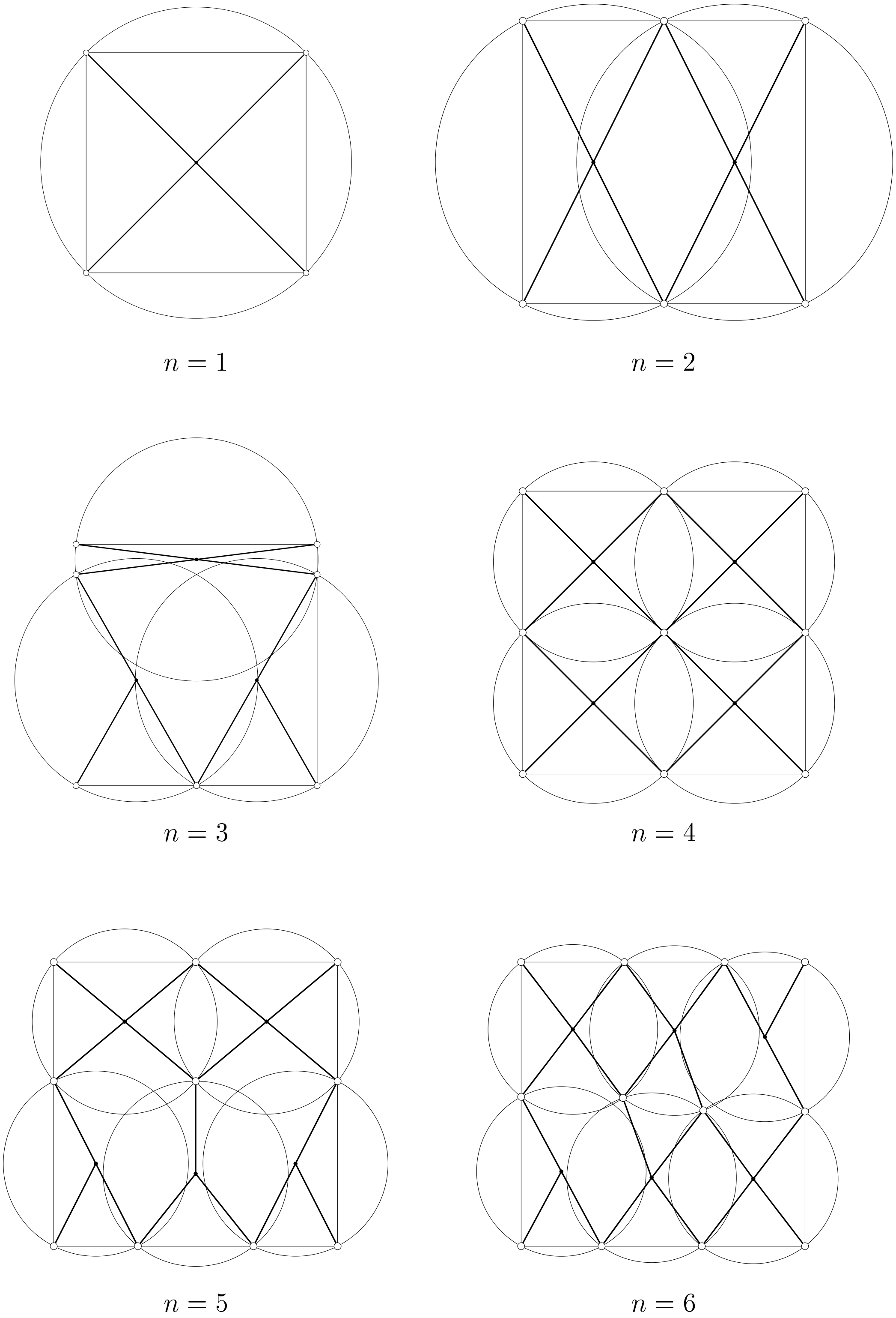


Figure 2: Coverings for $n = 1$ to $n = 6$.

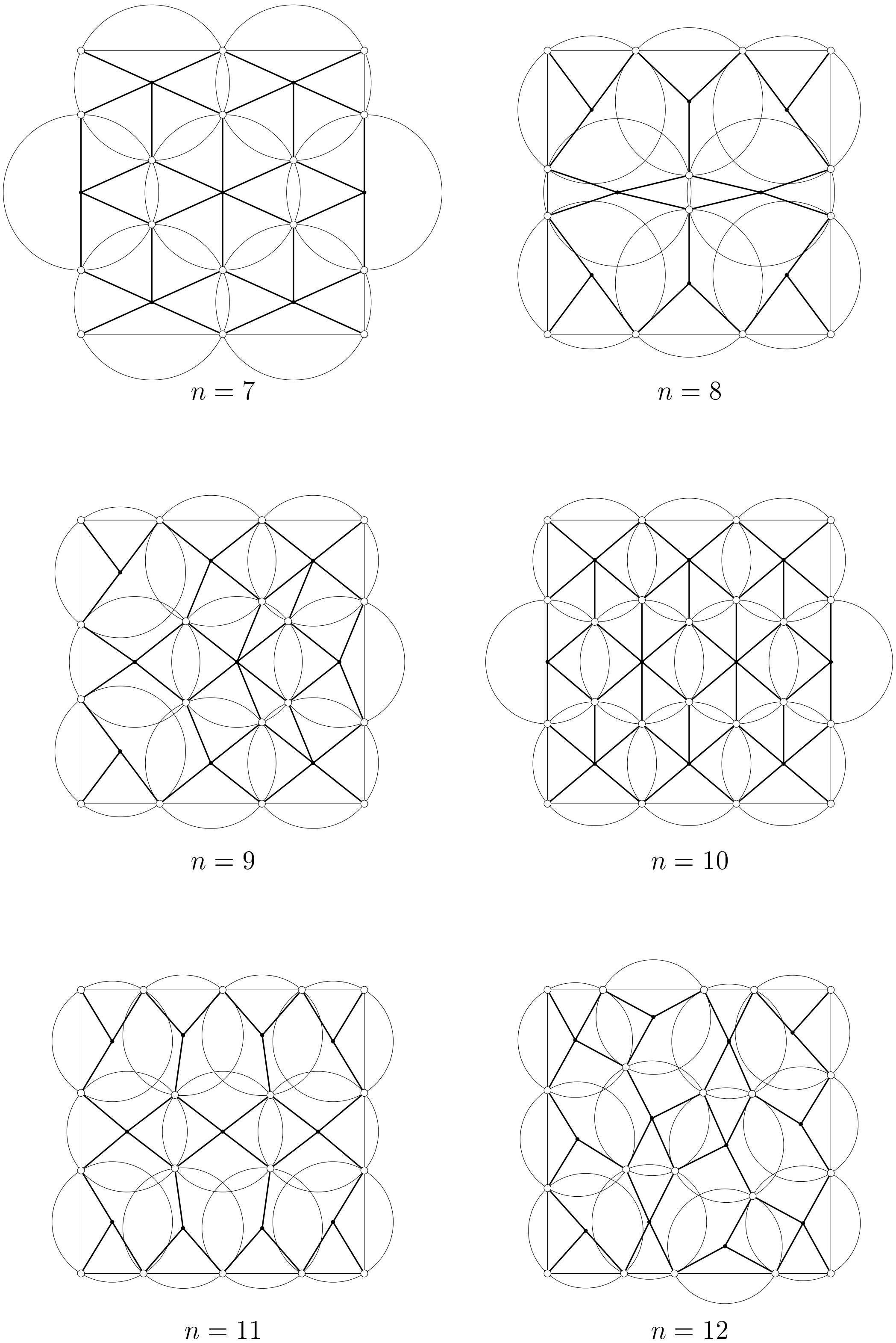
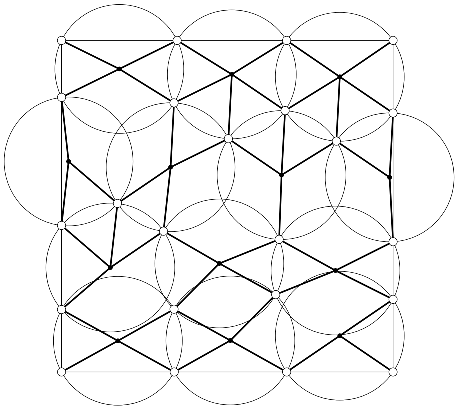
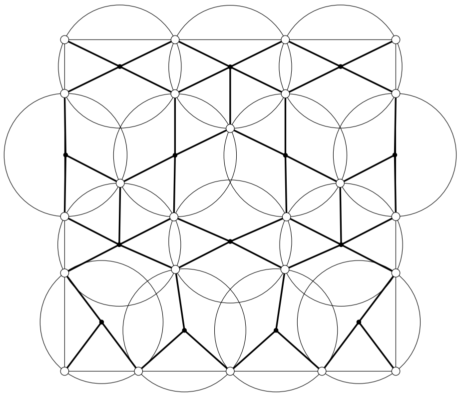


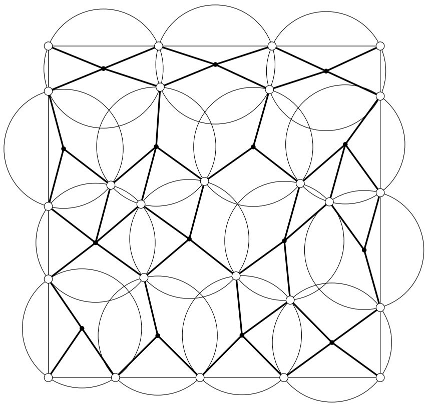
Figure 3: Coverings for $n = 7$ to $n = 12$.



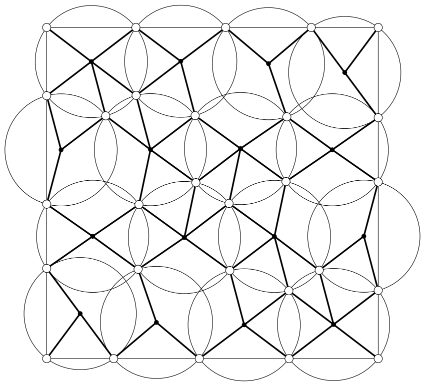
$n = 13$



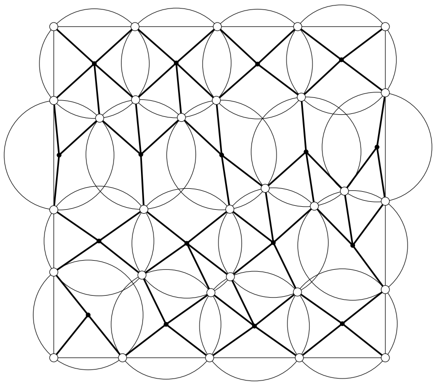
$n = 14$



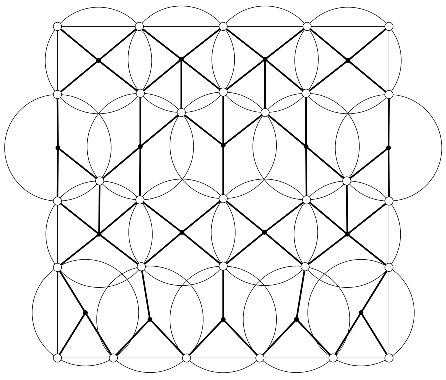
$n = 15$



$n = 16$



$n = 17$



$n = 18$

Figure 4: Coverings for $n = 13$ to $n = 18$.

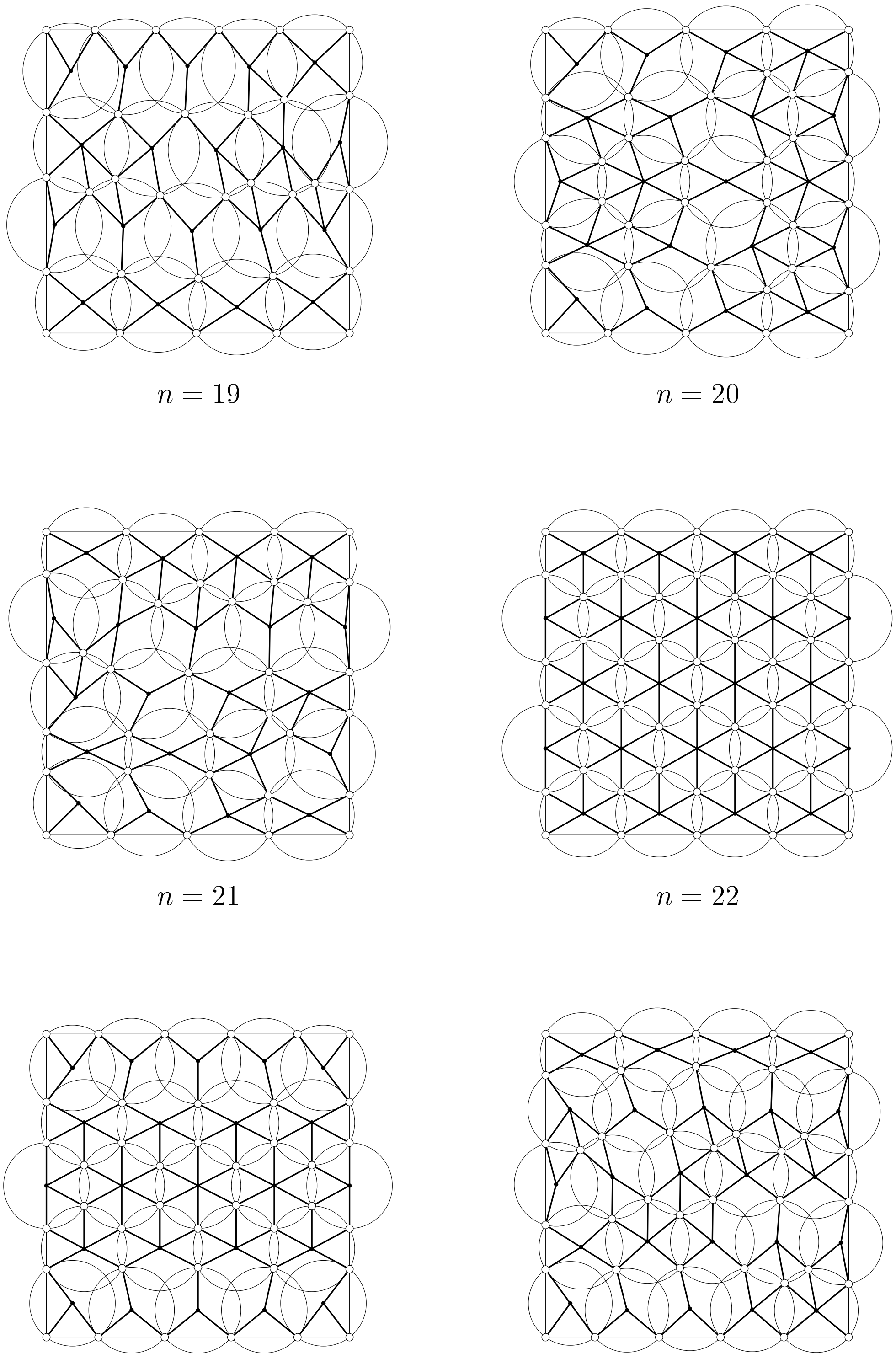


Figure 5: Coverings for $n = 19$ to $n = 24$.

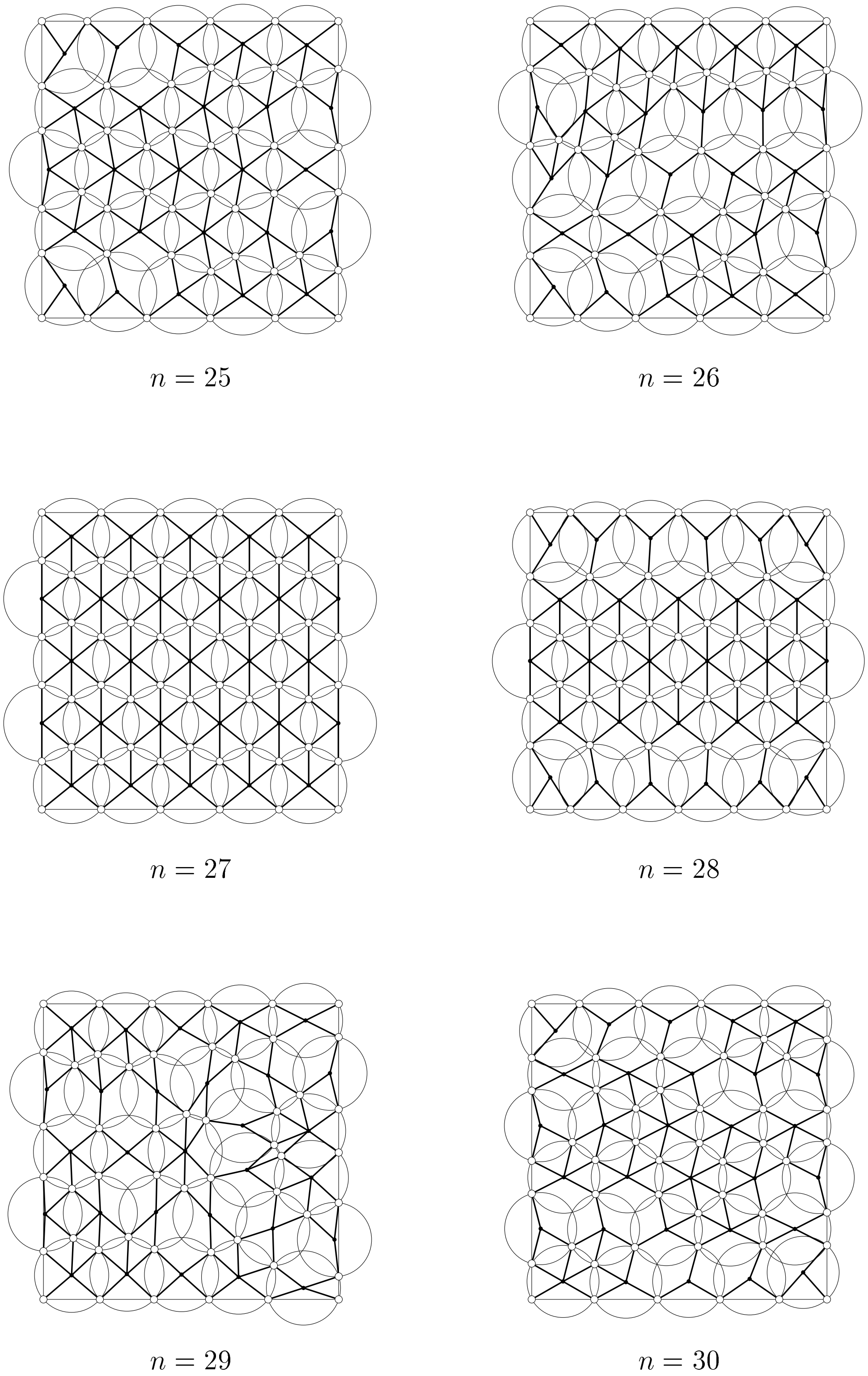


Figure 6: Coverings for $n = 25$ to $n = 30$.

Table 1: Properties of the coverings in Figures 2–6.

n	r_n	G	n	r_n	G
1	0.70710678118654752440...	D_4	16	0.16942705159811602395...	C_2
2	0.55901699437494742410...	D_2	17	0.16568092957077472538...	C_1
3	0.50389110926865935327...	D_1	18	0.16063966359715453523...	D_1
4	0.35355339059327376220...	D_4	19	0.15784198174667375675...	C_1
5	0.32616058400398728086...	D_1	20	0.15224681123338031005...	D_1
6	0.29872706223691915876...	C_2	21	0.14895378955109932188...	C_1
7	0.27429188517743176508...	D_2	22	0.14369317712168800049...	D_2
8	0.26030010588652494367...	D_2	23	0.14124482238793135951...	D_2
9	0.23063692781954790734...	D_1	24	0.13830288328269767697...	C_1
10	0.21823351279308384300...	D_2	25	0.13354870656077049693...	D_1
11	0.21251601649318384587...	D_2	26	0.13176487561482596463...	C_1
12	0.20227588920818008037...	C_2	27	0.12863353450309966807...	D_2
13	0.19431237143171902878...	C_1	28	0.12731755346561372147...	D_2
14	0.18551054726041864107...	D_1	29	0.12555350796411353317...	C_1
15	0.17966175993333219846...	C_1	30	0.12203686881944873607...	C_2

In Table 1 we tabulate some properties of the best known coverings. The radius of the circles is denoted by r_n and the symmetry group by G (the groups are of type C_n, cyclic groups of order n, and D_n, dihedral groups of order $2n$). Note that $D_1 \cong C_2$; we have used D_1 for reflectional symmetries and C_2 for rotational symmetries. In some cases the radius can be calculated symbolically, for example, $r_{22} = (3\sqrt{21} - 8)/40$, but in most cases the symbolic value is very difficult to obtain.

Most of the coverings in Figures 2–6 have symmetries. However, the best covering is not always the one with the nicest and most symmetric structure. For example, the symmetric covering with 13 circles in Figure 7 ($r = 0.19432288\ldots$) can be slightly improved by breaking the symmetry, see Figure 4.

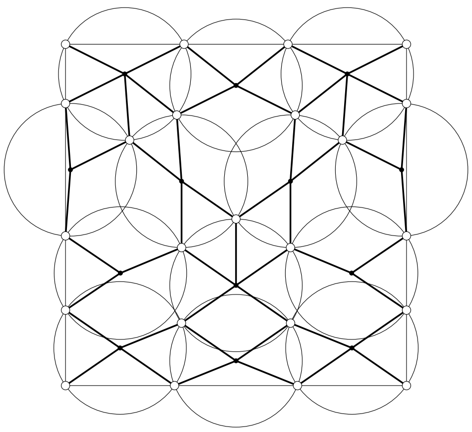


Figure 7: Symmetric covering of a square by 13 circles.

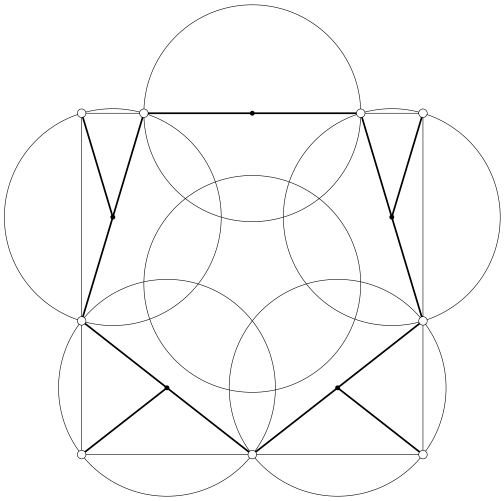


Figure 8: A locally optimal covering with a loose circle.

When *packing* circles in a square, a situation is often encountered where the positions of one or more circles are not exactly determined in the optimal packing [14, 16]. Circles that can move around in a limited area are called loose circles. Loose circles can also occur in locally optimal coverings in a square: one example is given in Figure 8 where we have a locally—not globally—optimal covering of a unit square with six circles (here $r = 0.3175623\ldots$). One of the circles can be moved freely in an area in the center of the square without exposing any of the square surface. The question whether any globally optimal covering of a square has loose circles remains open. None of the best coverings found in this study have loose circles, whereas several conjecturally optimal coverings of an equilateral triangle have at least one loose circle [9, 12].

The coverings with $n = 7$, 10, 22, and 27 have a very regular structure: the circles are aligned in straight rows, with k or $k + 1$ circles, the bottom and top row having k circles. It may be tempting to conjecture that these are members of infinite families of optimal coverings with $(2k^2 + 3k)/2$, k even, and $(2k^2 + k - 1)/2$, k odd, circles. Seemingly, the coverings with $(2k^2 + 3k + 2)/2$, k even, and $(2k^2 + k + 1)/2$, k odd, circles also make up families of good coverings.

In [15] it was conjectured that the circles in the optimal packing of $n + 1$ circles in a square always have smaller radius than those in the optimal packing of n circles. In view of the results in this work it seems that an analogous claim is true for circle coverings of a square; conjecturally, there exist no optimal coverings of a square with n and $n + 1$ circles such that the radius of the circles is the same.

ACKNOWLEDGEMENTS

Experiments with a covering program written by Ari Korhonen helped to develop the program used in this study. The financial support of the Academy of Finland is gratefully acknowledged.

References

[1] A. Ben-Israel, A Newton-Raphson method for the solution of systems of equations, *J. Math. Anal. Appl.* **15** (1966), 243–252.

[2] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991.

[3] D. M. Gay, Algorithm 611: Subroutines for unconstrained minimization using a model/trust-region approach, *ACM Trans. Math. Software* **9** (1983), 503–524.

[4] R. L. Graham, B. D. Lubachevsky, K. J. Nurmela, and P. R. J. Östergård, Dense packings of congruent circles in a circle, *Discrete Math.* **181** (1998), 139–154.

[5] R. H. Hardin, N. J. A. Sloane, and W. D. Smith, *Spherical Codes*, in preparation.

[6] A. Heppes and H. Melissen, Covering a rectangle with equal circles, *Period. Math. Hungar.* **34** (1997), 65–81.

[7] A. Lengyel and I. A. Veres, Egységnégyzet lefedése egybevágó körökkel (Covering the unit square with congruent circles), Competition essay, Technical University Budapest, Hungary, 1996.

[8] H. Melissen, Loosest circle coverings of an equilateral triangle, *Math. Mag.* **70** (1997), 119–125.

[9] H. Melissen, *Packing and Covering with Circles*, Ph.D. Thesis, Universiteit Utrecht, the Netherlands, 1997.

[10] J. B. M. Melissen and P. C. Schuur, Improved coverings of a square with six and eight equal circles, *Electron. J. Combin.* **3** (1996) R32, 10 pp. (electronic).

[11] J. B. M. Melissen and P. C. Schuur, Covering a rectangle with six and seven circles, *Discrete Appl. Math.* **99** (2000), 149–156.

[12] K. J. Nurmela, Conjecturally optimal coverings of an equilateral triangle with up to 36 equal circles, *Experiment. Math.*, to appear.

[13] K. J. Nurmela, Covering a circle by congruent circular discs, submitted for publication.

[14] K. J. Nurmela and P. R. J. Östergård, More optimal packings of equal circles in a square, *Discrete Comput. Geom* **22** (1999), 439–457.

[15] K. J. Nurmela and P. R. J. Östergård, Packing up to 50 equal circles in a square, *Discrete Comput. Geom.* **18** (1997), 111–120.

[16] R. Peikert, D. Würtz, M. Monagan, and C. de Groot, Packing circles in a square: A review and new results, in: P. Kall, ed., *System Modelling and Optimization: Proceedings of the 15th IFIP Conference Zürich, Switzerland, September 2–6, 1991*, Springer-Verlag, Berlin, 1992, 45–54.

[17] T. Tarnai and Zs. Gáspár, Covering a square by equal circles, *Elem. Math.* **50** (1995), 167–170.

[18] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer*, 2nd ed., Addison-Wesley, Redwood City, California, 1991.

[19] C. T. Zahn, Jr., Black box maximization of circular coverage, *J. Res. Nat. Bur. Standards Sect. B* **66** (1962), 181–216.

HELSINKI UNIVERSITY OF TECHNOLOGY LABORATORY FOR THEORETICAL COMPUTER SCIENCE
RESEARCH REPORTS

HUT-TCS-A49 Tomi Janhunen
Non-Monotonic Systems: A Framework for Analyzing Semantics and Structural Properties of NMR. March 1998.

HUT-TCS-A50 Ilkka Niemelä (Ed.)
Proceedings of the HeCSE Workshop on Emerging Technologies in Distributed Systems. March 1998.

HUT-TCS-A51 Kimmo Varpaaniemi
On the Stubborn Set Method in Reduced State Space Generation. May 1998.

HUT-TCS-A52 Ilkka Niemelä, Torsten Schaub (Eds.)
Proceedings of the Workshop on Computational Aspects of Nonmonotonic Reasoning. May 1998.

HUT-TCS-A53 Stefan Rönn
Semantics of Semaphores. 1998.

HUT-TCS-A54 Antti Huima
Analysis of Cryptographic Protocols via Symbolic State Space Enumeration. August 1999.

HUT-TCS-A55 Tommi Syrjänen
A Rule-Based Formal Model For Software Configuration. December 1999.

HUT-TCS-A56 Keijo Heljanko
Deadlock and Reachability Checking with Finite Complete Prefixes. December 1999.

HUT-TCS-A57 Tommi Junttila
Detecting and Exploiting Data Type Symmetries of Algebraic System Nets during Reachability Analysis. December 1999.

HUT-TCS-A58 Patrik Simons
Extending and Implementing the Stable Model Semantics. April 2000.

HUT-TCS-A59 Tommi Junttila
Computational Complexity of the Place/Transition-Net Symmetry Reduction Method. April 2000.

HUT-TCS-A60 Javier Esparza, Keijo Heljanko
A New Unfolding Approach to LTL Model Checking. April 2000.

HUT-TCS-A61 Tuomas Aura, Carl Ellison
Privacy and accountability in certificate systems. April 2000.

HUT-TCS-A62 Kari J. Nurmela, Patric R. J. Östergård
Covering a Square with up to 30 Equal Circles. June 2000.

ISBN 951-22-5072-1
ISSN 0783-5396