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New Upper Bounds for Binary/Ternary Mixed Covering Codes

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Abstract: A table of upper bounds for $K_{3,2}(n_1, n_2; R)$, the minimum number of codewords in a covering code with n_1 ternary coordinates, n_2 binary coordinates, and covering radius R, in the range $n = n_1 + n_2 \leq 13$, $R \leq 3$, is presented. Explicit constructions of codes are given to prove the new bounds and verify old bounds. These binary/ternary covering codes can be used as systems for the football pool game. The results include a new upper bound for the football pool problem for 9 matches, showing that $K_3(9, 1) \leq 1356$.

Keywords: Code construction, covering code, covering radius, football pool problem, mixed code, simulated annealing.

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Contents

1 Introduction

Packing and covering problems naturally arise within many areas of mathematics; much research has also been done on problems of these types (see, e.g., [29, 30]). In coding theory, most of the interest has been directed at the packing problem. This is very natural, since packing codes can be used for error-correcting purposes. Covering codes [6, 7, 10], on the other hand, have received much less attention. A covering code has the property that all words in a space are within a specified Hamming distance (the covering radius of the code) from at least one codeword.

The most important application of covering codes has turned out to be the design of football pool systems; the problem of constructing good covering codes is thus sometimes referred to as the football pool problem. In [12] a table of upper bounds for such binary/ternary mixed covering codes is presented. The references for many of the bounds in that table are to Finnish and Swedish football pool magazines, and are thus not easily verifiable; one of our intentions with this paper is to give proofs of such bounds. Moreover, our table contains as many as 64 new upper bounds. For further discussion on the application of mixed codes to football pool systems, see [12] and [32].

In Section 2 we introduce the notations needed and give an exact mathematical formulation of the problem. Although our aim is to construct good binary/ternary mixed codes, other types of mixed codes are needed in some of the constructions, so the mixed codes are discussed in a general sense. The rest of the section is devoted to construction methods and the determination of some exact values. Finally, an updated table of upper bounds is given in Section 3, which also contains extensive tables of codes, and the paper is concluded in Section 4.

2 Construction Methods

We consider an arbitrary *mixed space* $H = F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$, where $m \geq 1$, $n_i \geq 0$, $\sum_{i=1}^{m} n_i \geq 1$, and $q_i \geq 2$ (references to this space will be made throughout the paper). If not all coordinates are of the same arity, we say that the space is a *proper* mixed space. A code in such a space is called a *proper* mixed code. The *Hamming distance* $d(x, y)$ between two words $x, y \in H$ is the number of coordinates in which they differ; for $C \subseteq H$, $d(x, C) = \min\{d(x, c) \mid c \in C\}$. A code $C \subseteq H$ is now said to be a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; |C|)R$ code, where the *covering radius* R is defined as $\max_{x \in H} d(x, C)$. We further denote

$$K_{q_1,q_2,\ldots,q_m}(n_1,n_2,\ldots,n_m;R) = \min\{M \mid \text{there is a } (q_1,q_2,\ldots,q_m;n_1,n_2,\ldots,n_m;M)R \text{ code}\}.$$

A variety of notations has been used in the literature for special cases of this function, including $K_q(n,R)$ for the general nonmixed case ($m = 1$), $K(n,R)$ for the binary case ($q = 2$), $\sigma(n,k) = K_k(n,1)$, and $\sigma_n = K_3(n,1)$. The determination of σ_n is commonly referred to as the *football pool problem* [17]. For binary/ternary mixed codes, $K(t,b,R) = K_{3,2}(n_1,n_2;R)$ has been used in [12, 21].

Earlier considerations of mixed codes have mainly regarded constructions of perfect codes. Perfect codes are both packing and covering, i.e., for all $x \in H$, $|\{c \in C \mid d(x,c) \leq R\}| = 1$. References to this work can be found in [33].

In this paper we discuss binary/ternary mixed codes of length at most 13 and radius at most 3, i.e., we are interested in $K_{3,2}(n_1,n_2;R)$, $n_1+n_2 \leq 13$, $R \leq 3$. Very few exact values of this function are known, so efforts have been made to improve upper and lower bounds. We mainly consider upper bounds, which can be obtained by explicitly constructing covering codes. In some cases we prove the exact value of $K_{3,2}(n_1,n_2;R)$. We now present the methods that are used in the constructions.

2.1 Direct Constructions

In this section we present methods that can be used to construct new covering codes from given covering codes. Theorems 1 and 2 were proved in [27] and their corollaries—for the binary/ternary case—in [12].

Theorem 1
$K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R) \leq q_1 \cdot K_{q_1,q_2,\ldots,q_m}(n_1,n_2,\ldots,n_m;R)$.

Corollary 1 $K_{3,2}(n_1,n_2+1;R) \leq 2K_{3,2}(n_1,n_2;R)$.

Corollary 2 $K_{3,2}(n_1+1,n_2;R) \leq 3K_{3,2}(n_1,n_2;R)$.

Theorem 2 $K_{q_1,q_2,\ldots,q_m}(n_1,n_2+1,\ldots,n_m;R) \leq$

1. $K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R) + (q_2-q_1)\cdot\lfloor K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)/q_1\rfloor$, *if* $q_2 > q_1$,
2. $K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)$, *if* $q_2 \leq q_1$.

Corollary 3 $K_{3,2}(n_1+1, n_2; R) \leq \frac{3}{2}K_{3,2}(n_1, n_2+1; R)$.

In [27] attention was paid to the fact that a better bound than that in the first part of Theorem 2 can be obtained when the values of a q_1-ary coordinate of the code that the construction is acting on are not evenly distributed.

Theorem 3 *Let $C = C_0 \oplus \{0\} \cup C_1 \oplus \{1\}$ be a $(3,2; n_1, n_2+1; |C|)R$ code. Then $K_{3,2}(n_1+1, n_2; R) \leq 2|C_0| + |C_1|$.*

For $|C_0| < |C_1|$ Theorem 3 gives a better bound than Corollary 3. The following theorem was also proved in [27].

Theorem 4 *If q is a prime power and $k \geq 2$, then $K_{q_1=q, q_2=q^k, \ldots, q_m}(n_1 + (q^k-1)/(q-1), n_2, \ldots, n_m; R) \leq q^{(q^k-1)/(q-1)-k} \cdot K_{q_1=q, q_2=q^k, \ldots, q_m}(n_1, n_2+1, \ldots, n_m; R)$.*

This theorem can now be used to give an alternative proof of construction (e) in [12].

Corollary 4 $K_{3,2}(n_1, n_2+3; R) \leq \frac{8}{3}K_{3,2}(n_1+1, n_2; R)$.

Proof: $K_{3,2}(n_1, n_2+3; R) \leq 2K_{3,2,4}(n_1, n_2, 1; R)$ (Theorem 4)
$\leq \frac{8}{3}K_{3,2}(n_1+1, n_2; R)$ (Theorem 2). □

In some cases we can make use of perfect mixed covering codes to get good binary/ternary covering codes. The following theorem follows from the results in [20].

Theorem 5 *If q is a prime power, then there is a perfect covering of $F_{q^a}F_{q^b}^{n-1}$ $(a \geq b)$ with covering radius 1 iff there is an $r \geq a+b$, such that*

$$q^r = q^a + (n-1)(q^b - 1).$$

By applying Theorems 2 and 4 to perfect codes obtained from Theorem 5, we get some best known bounds for binary/ternary covering codes, as the following corollary shows.

Corollary 5 $K_{3,2}(1,4;1) \leq 8$, $K_{3,2}(1,12;1) \leq 1024$, $K_{3,2}(4,8;1) \leq 1296$.

Proof: $K_{4,2}(1,4;1) = 8$, so $K_{3,2}(1,4;1) \leq 8$. $K_{4,2}(1,12;1) = 1024$, so $K_{3,2}(1,12;1) \leq 1024$. $K_{8,2}(1,8;1) = 128$, so $K_{9,2}(1,8;1) \leq 144$, and then $K_{3,2}(4,8;1) \leq 1296$. □

Two bounds that can be obtained by applying the theorems and corollaries presented here to codes that are not record-holding binary/ternary codes (and thus a direct reference cannot be given in the main table in Section 3) are proved in the next theorem.

Theorem 6 $K_{3,2}(1, 11; 2) \leq 96$, $K_{3,2}(2, 8; 2) \leq 48$.

Proof: From Table I we know that $K_{4,3,2}(1, 1, 8; 2) \leq 48$. Theorem 4 and Theorem 2(2) can then be used to prove that $K_{3,2}(1, 11; 2) \leq 96$ and $K_{3,2}(2, 8; 2) \leq 48$, respectively. □

2.2 The Matrix Method

The idea behind the matrix method was first presented by Kamps and van Lint in [18]; it was later refined by Blokhuis and Lam in [1]. The method was originally used in constructions of codes with covering radius one, but it can easily be generalized to work for any covering radius; this has been independently shown by Carnielli [4] and van Lint Jr. [22]. We are now going to show how the method can be applied to mixed covering codes.

Let $\mathbf{A} = (\mathbf{I}; \mathbf{M}) = (a_1, \ldots, a_n)$ be an $r \times n$ matrix where $\mathbf{I}$ is the $r \times r$ identity matrix and $\mathbf{M}$ is a $r \times (n - r)$ matrix. We further define two functions $f_1 : \{1, 2, \ldots, r\} \rightarrow \{q_1, q_2, \ldots, q_m\}$ and $f_2 : \{1, 2, \ldots, n\} \rightarrow \{q_1, q_2, \ldots, q_m\}$ that associate arities to the rows and the columns of $\mathbf{A}$, respectively. Due to the $\mathbf{I}$ matrix part we have that for all $1 \leq i \leq r$,

$$a_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

and we require that $f_1(i) = f_2(i)$. To be exact, $a_{i,i}$ is the unit element in $F_{f_1(i)}$. For $1 \leq i \leq r$ and $r + 1 \leq j \leq n$ we require that $a_{i,j} \in F_{f_1(i)}$, if $f_1(i) = f_2(j)$ and that $a_{i,j} = 0$, if $f_1(i) \neq f_2(j)$.

Let $H = F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$ as defined earlier (so $f_1(1) = \cdots = f_1(n_1) = q_1$, etc.). For a matrix $\mathbf{A}$ fulfilling the earlier requirements we introduce the following distance function ($x, y \in H$):

$$d_A(x, y) = \min\{d \mid x + \sum_{j=1}^{d} \alpha_{i_j} a_{i_j} = y\},$$

where $1 \leq i_j \leq n$ and $\alpha_{i_j} \in f_2(i_j)$. We consider values $d \geq 0$, and let $\sum_{j=1}^{0} \cdots = (0, \ldots, 0)$. For $S \in H$ we define $S_{A,R}(s) = \{x \in H \mid d_A(s,x) \leq R\}$, and say that s R-covers $S_{A,R}(s)$ using $\mathbf{A}$. A set $S \in H$ *R-covers H using* $\mathbf{A}$ if

$$\bigcup_{s \in S} S_{A,R}(s) = H.$$

Let $G = F_{f_2(1)} F_{f_2(2)} \cdots F_{f_2(n)}$, the length of which is n. The kernel of the matrix method is the result presented in the following lemma.

Lemma 1 *Let $x, y \in H$, and take any $y' \in G$ such that $Ay' = y$. Then there is an $x' \in G$, such that $Ax' = x$ and $d(x', y') = d_A(x, y)$.*

Proof: We have that $x = y + \sum_{j=1}^{d_A(x,y)} \alpha_{i_j} a_{i_j} = Ay' + At = A(y' + t)$, where $t \in G$ has $d_A(x,y)$ nonzero coordinates. Now $x' = y' + t$ fulfills $Ax' = x$ and $d(x', y') = d_A(x, y)$. $\square$

Using Lemma 1, the next theorem (which is also presented in [27]) follows immediately.

Theorem 7 *If S R-covers H using an $r \times n$ matrix $\mathbf{A} = (\mathbf{I}; \mathbf{M})$, then $W = \{w \in G \mid \mathbf{A}w \in S\}$ R-covers G. $|W| = |S| \cdot \prod_{i=r+1}^{n} f_2(i)$.*

This method has earlier successfully been used to find many good (nonmixed) covering codes [1, 19, 27, 28]; these papers can also be consulted for more detailed information about how to use the method in practise. When we have found a code using the matrix method, we have tried to simplify its presentation as much as possible, i.e., we have tried to find a covering with as few words as possible in S. However, we did not spend very much time on this task, so some of the codes can probably still be written in a simpler form.

2.3 Partitioning Constructions

Partitioning constructions exist for many types of codes. The idea behind this sort of constructions is that good new codes are obtained whenever the codewords of existing codes can be partitioned into subcodes with certain properties. We first define the subnorm and the norm of mixed codes. Codes with these partitioning properties are used to construct many of the new codes.

Definition 1 *Suppose C is a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code. Then C has (k,t)-subnorm (S, R') if there is a partition of C into k nonempty subsets $C_0, C_1, \ldots, C_{k-1}$ such that $\min_a d(x, C_a) + \max_a d(x, C_a) \leq S$ whenever $R - t \leq d(x, C) \leq R$, and for all x, $\max_a d(x, C_a) \leq R'$.*

Such a partition is called acceptable. *If the subnorm is given as a single value and not as tuple, it refers to the S parameter. If C has (k,t)-subnorm $2R+1$, then it is called* (k,t)-subnormal. *If C is not (k,t)-subnormal, then it is called* (k,t)-absubnormal. *If C is $(k,0)$-subnormal, then it is called* k-seminormal. *If C is $(2R+1, R+1)$-subnormal, then it is called* strongly k-seminormal.

Let C be a $(q_1, \ldots, q_m; n_1, \ldots, n_m; M)R$ code with $C_a^{(i)\prime} = \{(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \mid (c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) \in C\}, a \in F_q$, where F_q is the arity of the i^{th} coordinate of C. The sets $C_a^{(i)\prime}$ are assumed to be nonempty for all a.

Definition 2 *The code C has (q,t)-norm $(N^{(i)}, R_i')$ with respect to the i^{th} coordinate if $\min_a d(x, C_a^{(i)\prime}) + \max_a d(x, C_a^{(i)\prime}) \leq N^{(i)}$ whenever $R - t \leq d(x, \bigcup_a C_a^{(i)\prime}) \leq R$, and for all x, $\max_a d(x, C_a^{(i)\prime}) \leq R_i'$. If $N^{(i)} \leq N, R_i' \leq R'$ for at least one coordinate i, we say that C has* (q,t)-norm (N, R') *(or N alone, cf. the definition of (q,t)-subnorm), and the coordinates i for which this equation holds are called* acceptable. *If C has (q,t)-norm $2R$ it is called* (q,t)-normal. *If C is not (q,t)-normal it is called* (q,t)-abnormal.

Although the theory of normal and subnormal codes is quite young, some old results on certain partitions of codes exist. In [9] Golomb and Posner considered $w(n,q)$, the maximum number k, such that the space $(q, n, q^n)0$ has $(k,0)$-subnorm $(1,1)$ (using our terminology) and they also give acceptable partitions. The code (space) is thus strongly seminormal for such k. The concept was generalized in [3], where $\delta(n,q,R)$, the maximum number k, such that the space $(q, n, q^n)0$ has $(k,0)$-subnorm (R,R), is treated. The problem of determining the values of these functions has been referred to as the *coloring problem*. If the codewords that belong to the same partition are "painted" with the same color, then any word has codewords with all possible colors within Hamming distance R.

In the *amalgamated direct sum* (ADS) construction, a code is partitioned according to the values of a given coordinate. Let A be a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M_A)R_A$ code and let B be a $(q_1', q_2', \ldots, q_{m'}'; n_1', n_2', \ldots, n_{m'}'; M_B)R_B$ code with $B = B_0 \cup \cdots \cup B_{q_m - 1}$ $(n_m, n_1' > 0)$. B_i is assumed to be nonempty for all i. We then define the ADS of A and B as

$$C = A \dot{\oplus} B = \bigcup_{i=0}^{q_m - 1} \{(a, b) \mid (a, i) \in A, b \in B_i\},$$

which is a $(q_1,\ldots,q_{m-1},q_m,q'_1,\ldots q'_{m'};n_1,\ldots,n_{m-1},n_m-1,n'_1,\ldots,n'_{m'};M)R_C$ code with $M=\sum_{i=0}^{q_m-1}|A_i^{(n)}|\cdot|B_i|$ codewords. For a proof of the next theorem, see [27, Theorem 20].

Theorem 8 *If B is strongly q_m-seminormal such that $B=B_0\cup\cdots\cup B_{q_m-1}$ is an acceptable partition, then $R_C\leq R_A+R_B$.*

A strongly 3-seminormal $(3,2;4,1;18)1$ code (see Tables III and V) and a strongly 4-seminormal $(3,4,12)1$ code (see [12, Construction i]) can be used as B in Theorem 8 to obtain the following theorems. The binary/ternary/quaternary codes that Theorem 10 has been applied to in order to obtain new upper bounds for codes in the main table are given in Table I. One of the bounds that can be obtained, $K_3(10,3)\leq 108$, was derived in another way in [25], however, that construction involved a seminormal code, the acceptable partition of which was not given in the paper.

Theorem 9 *If $n_1>0$, then $K_{3,2}(n_1+3,n_2+1;R+1)\leq 6K_{3,2}(n_1,n_2;R)$.*

Theorem 10 $K_{4,3,2}(n_1-1,n_2+4,n_3;R+1)\leq 3K_{4,3,2}(n_1,n_2,n_3;R)$.

Table I. $K_{4,3,2}(n_1,n_2,n_3;R)$.

n_1	n_2	n_3	R	Upper bound	Reference
1	0	4	1	8	Theorem 5
1	1	6	2	18	Table III
1	1	8	2	48	Table IV
1	2	3	2	8	Table III
1	3	4	2	30	Table IV
1	4	3	2	36	Table IV
1	6	0	2	36	Proof of Theorem 16

Corollary 6 *If $n_1>0$, then $K_{3,2}(n_1+3,n_2;R+1)\leq 4K_{3,2}(n_1,n_2;R)$.*

Proof: $K_{3,2}(n_1+3,n_2;R+1)\leq 3K_{4,3,2}(1,n_1-1,n_2;R)$ (Theorem 10) $\leq 3\cdot\frac{4}{3}K_{3,2}(n_1,n_2;R)$ (Theorem 2(1)) $=4K_{3,2}(n_1,n_2;R)$. □

If the last coordinate of A is binary, the following well-known result can be proved [7, Theorem 11].

Theorem 11 *If $q_m=q'_1=2$, if A is $(2,1)$-normal with its last coordinate acceptable, and if B is $(2,1)$-normal with its first coordinate acceptable, then $R_C\leq R_A+R_B$.*

Theorem 11 is used to obtain many bounds in Table II by letting one of the codes be a binary Hamming code. These are known to be $(2,1)$-normal. These Hamming codes are combined with codes from the table, which have been proved to be $(2,1)$-normal by a simple computer check.

By using Theorem 8 we obtain some record-breaking codes in the next theorem. Codes that are used and partitions of codes are listed in Tables III, IV and V.

Theorem 12 $K_{3,2}(3,8;1) \leq 576$, $K_{3,2}(3,10;1) \leq 2080$, $K_{3,2}(7,4;1) \leq 2304$, $K_{3,2}(8,2;1) \leq 1728$, $K_{3,2}(8,2;3) \leq 72$, $K_3(9,1) \leq 1356$.

Proof: $\mathbf{K_{3,2}(3,8;1) \leq 576}$: Take the ADS of a $(6,3,2;1,1,6;96)1$ code and a $(3,2;2,2;36)0$ code with $(6,0)$-subnorm $(1,1)$. This code has $16 \cdot 36 = 576$ codewords and covering radius 1.
$\mathbf{K_{3,2}(3,10;1) \leq 2080}$: Take the ADS of a $(8,2;1,8;128)1$ perfect code (cf. Theorem 5) and a $(3,2;3,2;130)0$ code with $(8,0)$-subnorm $(1,1)$. This code has $16 \cdot 130 = 2080$ codewords and covering radius 1.
$\mathbf{K_{3,2}(7,4;1) \leq 2304}$: Take the ADS of a $(6,3;1,6;288)1$ code and a $(3,2;1,4;48)0$ code with $(6,0)$-subnorm $(1,1)$. This code has $48 \cdot 48 = 2304$ codewords and covering radius 1.
$\mathbf{K_{3,2}(8,2;1) \leq 1728}$: Take the ADS of a $(6,3;1,6;288)1$ code and a $(3,2;2,2;36)0$ code with $(6,0)$-subnorm $(1,1)$. This code has $48 \cdot 36 = 1728$ codewords and covering radius 1.
$\mathbf{K_{3,2}(8,2;3) \leq 72}$: From Table II we know that $K_{3,2}(4,4;2) \leq 24$. It is easily shown that we can encode two binary coordinates into one 4-ary coordinate and retain the covering radius (see proof of [12, Construction h]). Thus $K_{4,3,2}(1,4,2) \leq 24$, and Theorem 10 gives $K_{3,2}(8,2;3) \leq 72$.
$\mathbf{K_3(9,1) \leq 1356}$: Take the ADS of a $(5,3;1,6;252)1$ code and a $(3,3,27)0$ code with $(6,0)$-subnorm $(1,1)$ (from [9, Fig. 7]). Then we can obtain a ternary code of length 9 and covering radius 1, which has $2 \cdot 6 \cdot 48 + 5 \cdot 48 + 2 \cdot 5 \cdot 54 = 1356$ codewords. □

The bound $K_3(9,1) \leq 1356$ is very interesting as it gives a new upper bound for the football pool problem for 9 matches. The following theorem (see [26, Theorem 1], [27, Theorem 17]) shows how codes can be combined using the *blockwise direct sum* (BDS), introduced by Honkala in [15].

Theorem 13 *Let A be a $(q_1,q_2,\ldots,q_m;n_1,n_2,\ldots,n_m;M_A)R_A$ code, and let B be a $(q'_1,q'_2,\ldots,q'_{m'};n'_1,n'_2,\ldots,n'_{m'};M_B)R_B$ code. Assume further that A has (k,R_A)-subnorm (S_A,R'_A) with the partition $A = A_1 \cup \cdots \cup A_k$ acceptable, and that B has (k,t_B)-subnorm S_B with the partition $B = B_1 \cup \cdots \cup B_k$ acceptable. Then the* blockwise direct sum *of A and B*

$$BDS(A,B) = \bigcup_{i=1}^{k} (A_i \oplus B_i)$$

is a $(q_1, \ldots, q_m, q'_1, \ldots q'_{m'}; n_1, \ldots, n_m, n'_1, \ldots, n'_{m'}; M)R$ *code with* $M = \sum_{i=1}^{k} |A_i| \cdot |B_i|$ *and* $R \leq \max\{R_1, R_2\}$*, where*

$$\begin{array}{rcl} R_1 & = & \lfloor (S_A + S_B)/2 \rfloor \\ R_2 & = & \begin{cases} 0 & \textit{if } t_B \geq R_B \\ R_B - t_B - 1 + R'_A & \textit{otherwise.} \end{cases} \end{array}$$

By using a punctured optimal code code as code A in Theorem 13, the following result can be obtained. The result has been presented in [13, Theorem 3] for nonmixed codes. The proof in the mixed case is essentially the same and is omitted.

Theorem 14 *If there is a* (q, R')*-normal* $(q, q'_1, \ldots, q'_{m'}; 1, n'_1, \ldots, n'_{m'}; M')R'$ *code* A *with the first coordinate acceptable and a* $(q, R'-1)$*-subnormal* $(q_1, \ldots, q_m; n_1, \ldots, n_m; M)R$ *code* B*, then* $K_{q_1,\ldots,q_m,q'_1,\ldots,q'_m}(n_1, \ldots, n_m, n'_1, \ldots, n'_{m'}; R + R') \leq M'M/q$.

Now, for example, any nontrivial perfect code can be used as code A in Theorem 14, since such codes have been proved to be (q, R)-normal [13, Lemma 1]. In the table of record-breaking codes the codes used as code A in this construction are the following: 1) the $(2, 3, 2)1$ Hamming code, 2) the $(3, 4, 9)1$ Hamming code, and 3) a $(4, 2; 1, 4; 8)1$ perfect code, 4) the following $(2, 1)$-normal $(3, 2; 1, 2; 3)1$ code: $\{000, 111, 211\}$ (puncturing this in the last coordinate we get a partition $\{00\} \cup \{11\} \cup \{21\}$), and 5) the following $(3, 1)$-normal $(3, 3, 5)1$ code: $\{000, 111, 122, 212, 221\}$ (puncturing this we get a partition $\{00\} \cup \{11, 22\} \cup \{12, 21\}$).

Whereas the calculation of (k, t)-norms for codes is a straight-forward task, (good) (k, t)-subnorms are not generally easily calculated. In some special cases, good (k, t)-subnorms can be found by combinatorial arguments [14]. For the general case, the simulated annealing-based approach presented in [26] can be used. Acceptable partitions of codes, which are used in Theorem 14 to construct best known codes, are given in Table V; except for a (3,6,18)2 code and the $(3, 11, 729)2$ Golay code. A generator matrix for the Golay code is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 \end{bmatrix}.$$

The partition $C = C_0 \cup C_1 \cup C_2$, where C_i consists of codewords with parity-check sum i, proves that C is $(3,2)$-subnormal (easily checked by computer).

It can be shown that for codes obtained by the matrix method, we can partition the set S with distance function $d_A(x,y)$ when we want to calculate a good subnorm for the code. Unfortunately, it has turned out that in many cases where an acceptable partition can be found when all codewords are written out, no acceptable partition can be found of the words in S.

2.3.1 Some Special Cases

In [26] some special cases for partitioning constructions were considered. It was first shown how near-coverings can be used to construct good codes; this was used to prove that $K_{3,2}(3,10;2) \leq 358$. The $(2,10,118)2$ code involved was not given in that paper. In Table IV we list such a code (in matrix form) and in Table V an acceptable partition (for the construction in [26, Corollary 4]) is given.

Another special case described in [26] is the case when a code is nearly (k,t)-subnormal. In [26, Corollary 5] it was shown how such a code can be combined with a punctured $(3,4,9)1$ Hamming code. In this way a record-breaking $(3,2;5,4;64)2$ was constructed. We can also use a punctured $(2,3,2)1$ code as code $B = B_1 \cup B_2 = \{00\} \cup \{11\}$ in [26, Theorem 5] to obtain the following result.

Theorem 15 *Let $A = A_1 \cup A_2$ be a $(3,2;n_1,n_2;M)R_A$ code ($R_A \geq 1$), and let $S_a = \{x \in F_q^{n_A} \mid d(x,A) + d(x,A_a) > 2R_A + 1, d(x,A) = R_A\}$. $S = \cup S_a$. Further assume that C_a covers S_a with covering radius $R_A + 1$. Then*

$$K_{3,2}(n_1, n_2 + 2; R_A + 1) \leq M_A + \sum_{a=1}^{2} |C_a|.$$

No absubnormal codes with binary coordinates are known, so the cases where we can make use of Theorem 15 in the construction of binary/ternary codes are restricted to constructions starting from ternary codes. The ternary Hamming code of length 4 is not $(3,1)$-subnormal [14]; it has turned out that it is not even $(2,1)$-subnormal.

Example 1. The ternary Hamming code of length 4 can be partitioned as follows: $A = A_1 \cup A_2 = \{0201, 1210, 2011, 2100, 2222\} \cup \{0020, 0112, 1002, 1121\}$. Then $S_1 = \phi$ and $S_2 = \{2200, 2202, 2210, 2211\}$, so by choosing $C_2 = \{2222\}$ we obtain $K_{3,2}(4,2;2) \leq 9+1 = 10$.

Codes of this type can generally be found by simply combining codes to get a near-covering, and then try to cover the uncovered words using as few codewords as possible. This is illustrated by the following three examples.

Example 2. The $(3,2;5,3;168)1$ code in Table III is almost 4-seminormal (a partition is given in Table V). The ADS of this code and a $(4,2;1,4;8)1$ code gives a $(3,2;5,7;336)R$ code that covers all but 36 words with covering radius 2. It is no difficult task to find 12 codewords that cover these words with covering radius 2, so $K_{3,2}(5,7;2) \leq 336+12 = 348$.

Example 3. The $(3,2;7,1;54)2$ code listed in Table III is almost 3-seminormal (again, a partition is given in Table V). The ADS of this code and the $(3,4,9)1$ Hamming code gives a $(3,2;10,1;162)R$ code that covers all but 162 words with covering radius 3. A covering can be obtained by adding 27 codewords, so $K_{3,2}(10,1;3) \leq 162+27 = 189$.

Example 4. The $(3,3,5)1$ code $C = C_1 \cup C_2 \cup C_3 = \{000\} \cup \{111, 122\} \cup \{212, 221\}$ is almost 3-seminormal; one possible partition is mentioned above. The ADS of C and a $(3,2;4,4;24)2$ code gives a $(3,2;6,4;40)R$ code that covers all but 68 words with covering radius 3. There are 4 codewords that cover these words with covering radius 3, so $K_{3,2}(6,4;3) \leq 40+4 = 44$.

2.4 Combinatorial Constructions

In this subsection we discuss codes obtained by combinatorial constructions. Some of the bounds (Theorem 16) are given by codes that are constructed case by case. Actually, combinatorial proofs of more bounds could be given, however, we have stopped at a point were the proofs become too long and inelegant.

Theorem 16 $K_{3,2}(1,11;1) \leq 548$, $K_{3,2}(5,2;1) \leq 96$, $K_{3,2}(5,3;1) \leq 168$, $K_{3,2}(6,2;2) \leq 48$, $K_{3,2}(6,3;2) \leq 72$.

Proof: $\mathbf{K_{3,2}(1,11;1) \le 548}$: Let S be the 66 words in the Steiner system $S(4,5,11)$. Then $C' = \{0,1,2\} \oplus (S \cup \overline{S})$ covers all words with 4 to 7 ones in the binary coordinates. Now we partition the binary coordinates into four sets (S_1, S_2, S_3, and S_4) with 3,3,3, and 2 coordinates, respectively. Let C_0 (C_1, C_2) be the binary code of length 11 with all codewords of weight 2 such that both ones belong either to $S_1 \cup S_2$ or $S_3 \cup S_4$ ($S_1 \cup S_3$ or $S_1 \cup S_4$, $S_1 \cup S_4$ or $S_2 \cup S_3$). We now show that $C'' = \{0\} \oplus C_0 \cup \{1\} \oplus C_1 \cup \{2\} \oplus C_2$ covers all words with 1, 2, or 3 ones in the binary coordinates with covering radius 1. The case with 2 ones is clear, since all binary words of weight 2 belong to some code C_i. A word (a,x), $a \in F_3$, $x \in F_2^1 1$, $wt(x) = 1$ is covered since there is a word $y \in C_a$, such that $d(x,y) = 1$. A word (a,x), $a \in F_3$, $x \in F_2^{11}$, $wt(x) = 3$ is covered since at least two of the three coordinates with ones must belong to the same subset of coordinates $S_i \cup S_j$, so $d(x, C_a) = 1$. Let C''' be the code obtained from C'' by complementing the binary part. Now $C' \cup C'' \cup C''' \cup (0,0,\ldots,0) \cup (0,1,\ldots,1)$ is a $(3,2;1,11;M)1$ code with $M = 2 \cdot (3 \cdot 66 + 3 \cdot (\binom{6}{2} + \binom{5}{2}) + 1) = 548$ codewords.

$\mathbf{K_{3,2}(5,2;1) \le 96}$: We start from the codes obtained by cyclic shifts of the following ternary words: C_0 : $01221, 12002, 20110, 00000, 11111, 22222$ (18 words), C_1 : $02112, 10220, 21001, 00000, 11111, 22222$ (18 words), C_2 : $00022, 00101, 00111, 02022, 11212, 11222$ (30 words), and C_3 : $00011, 00202, 00222, 01011, 22121, 22111$ (30 words). Now, e.g., $C_2 \cup C_3$ is a partition of all words with three occurrences of one value and two of another. The following property of these codes can now relatively easily be verified (again, symmetries help a lot). If a word $x \in F_3^5$ is not within distance 1 from a word in C_0 (C_1, C_2, or C_3), then $x \in C_1 \cup C_2$ ($C_0 \cup C_3$, C_0, or C_1, respectively). Then it follows that $C = C_0 \oplus \{00\} \cup C_1 \oplus \{01\} \cup C_2 \oplus \{10\} \cup C_0 \oplus \{11\}$ is a $(3,2;5,2;96)1$ code.

$\mathbf{K_{3,2}(5,3;1) \le 168}$: We use the codes C_i defined in the proof of $K_{3,2}(5,2;1) \le 96$ and let $C = C_0 \oplus \{001, 010, 100\} \cup C_1 \oplus \{011, 101, 110\} \cup C_2 \oplus \{000\} \cup C_3 \oplus \{111\}$. C is then a $(3,2;5,3;M)1$ code with $M = 2 \cdot 3 \cdot 18 + 2 \cdot 30 = 168$ codewords.

$\mathbf{K_{3,2}(6,3;2) \le 72}$: Let $A = \{012, 120, 201\}$ and $B = \{021, 102, 210\}$. Further let $C_0 = \{000, 111\} \oplus A \cup \{222\} \oplus B$, $C_1 = \{000, 111\} \oplus B \cup \{222\} \oplus A$, $C_2 = A \oplus \{000, 111\} \cup B \oplus \{222\}$, and $C_3 = B \oplus \{000, 111\} \cup A \oplus \{222\}$. We now show that for all $x \in F_3^6$ and $i \in F_4$, if x is not within distance 2 from C_i, then there is a $j \in F_4$, such that x is within distance 1 from C_j. Due to symmetries we can set $i = 0$. Words that are at distance at least 3 from C_0 are of form (u,v), $u, v \in F_3^3$, where u and v are of one of the following forms: a) Two coordinates of u have the same values, all coordinates of v have the same values: Now $(A \cup B) \oplus \{v\} \subset C_2 \cup C_3$, so either $d((u,v), C_2) = 1$ or $d((u,v), C_3) = 1$. b) All coordinates of u have distinct values, two coordinates of v have the same values: Now $\{u\} \oplus \{000, 111, 222\} \subset C_2 \cup C_3$, so either $d((u,v), C_2) = 1$ or $d((u,v), C_3) = 1$. c) All coordinates of u have

distinct values, all coordinates of v have the same values: See b), now either $d((u,v),C_2) = 0$ or $d((u,v),C_3) = 0$. d) None of the coordinates of u has value 2, $u \notin \{000, 111\}$, $v \in B$: Now $d((u,v),C_1)) = 1$. From this property of the sets C_i it is easy to deduce that $C = \cup_{i=0}^{3}\{i\} \oplus C_i$ is a $(4,3;1,6;36)2$ code. We finally apply Theorem 4 to obtain a $(3,2;6,3;72)2$ code that proves the bound.
$\mathbf{K_{3,2}(6,2;2) \leq 48}$: We first shorten the previously obtained $(3,2;6,3;72)2$ code in one binary coordinate to obtain $C' = C_0 \oplus \{00\} \cup C_1 \oplus \{11\} \cup C_2 \oplus \{10\} \cup C_3 \oplus \{01\}$ (the sets C_i are from the proof of $K_{3,2}(6,3;2) \leq 72$). C' covers all but 72 words in $F_3^6 F_2^2$ with covering radius 2. E.g., a word $(u,(1,1))$, $u \in F_3^6$ is not covered for $u \in \{001, 010, 011, 100, 101, 110\} \oplus A$. These words can be covered by the following set of three codewords: $\{000\} \oplus A \oplus \{11\}$. Due to symmetries we can now construct a $(3,2;6,2;M)2$ code with $M = 36 + 4 \cdot 3 = 48$ codewords. □

In [27] another argumentation was used to show that $K_{3,2}(5,2;1) \leq 96$. We finally give a more general combinatorial construction.

Theorem 17 *If there is a $(2,n_1,M_1)R_1$ code C', a $(2,n_2,M_2)R_2$ code C'', and $R_1 \leq R_2 - (n_2 - n_1 - 2)$, then $K_{3,2}(n_2 - n_1 + 1, n_1; R_2 + 1) \leq (M_1 + M_2)$.*

Proof: We show that the code $C = (C' \oplus (2,2,\ldots,2)) \cup C'''$, where the all-2 vector is of length $n_2 - n_1 + 1$ and C''' is obtained from C'' by adding an overall parity check bit, proves the upper bound. Consider any word $u \in F_2^{n_1} F_3^{n_2-n_1+1}$. If the values of at least 2 of the last $n_2 - n_1 + 1$ coordinates of u are 2, then $d(u,C) \leq R_1 + (n_2 - n_1 + 1 - 2) = R_1 + (n_2 - n_1 - 2) + 1 \leq R_2 + 1$. Otherwise the value of at most one of the last $n_2 - n_1 + 1$ coordinates is 2, so the values of these coordinates (except for at most one) belong to F_2. We then get that $d(u,C) \leq R_2 + 1$, since (according to the discussion preceding the theorem) the deletion of one of the last $n_2 - n_1 + 1$ coordinates of C''' gives a $(2,n_2,M_2)R_2$ code. □

Example 5. Let C' be a $(2,7,16)1$ code and let C'' be a $(2,10,30)2$ code. Then $K_{3,2}(4,7;3) \leq 46$ according to Theorem 17, whereas the best known code with these parameters proves that $K_{3,2}(4,7;3) \leq 45$.

2.5 Exact Values

In this paper we do not discuss lower bounds, however, it is interesting to know in what cases the upper bound and the exact value coincide. For some bounds this simply follows from the *sphere covering bound*,

$$K_{3,2}(n_1, n_2; R) \geq 3^{n_1} 2^{n_2} / \sum_{i=0}^{R} \sum_{j=0}^{i} \binom{n_1}{j} \binom{n_2}{i-j} 2^j.$$

This bound has been improved in [21], and has reached the constructive upper bound for some parameters (e.g., $K_{3,2}(3,3;1) \geq 24$, whereas according to the sphere covering bound, $K_{3,2}(3,3;1) \geq 22$). Exact values of $K_{3,2}(n_1, n_2; R)$ for nonmixed binary ($n_1 = 0$) and ternary ($n_2 = 0$) codes have been gathered from [7] and [5], respectively.

In many cases the sphere bound and the improved results in [21] can be refined using combinatorial arguments, as the following theorem shows.

Theorem 18 $K_{3,2}(1,3;1) = 6$.

Proof: Assume that $K_{3,2}(1,3;1) \leq 5$. We try to find 5 codewords that cover $F_3 F_2^3$. Then there is at least one ternary value which appears in the first coordinate in at most one codeword (it is easily seen that all values must appear in at least one codeword). W.l.o.g., assume that this value is 0, and that 0000 is a codeword. By considering the subspace $(0, x)$, $x \in F_2^3$, we see that the remaining codewords must be of form $a110$, $b101$, $c011$, and $d111$ ($a, b, c, d \in \{1, 2\}$). Now, by investigating how these cover (u, v), $u \in \{1, 2\}$, $v \in F_2^3$, $wt(v) = 1$, we get that $a \neq b$, $b \neq c$, and $c \neq a$, a contradiction. This completes the proof, since from Table II we know that $K_{3,2}(1,3;1) \leq 6$. □

Exact values can also be found by an exhaustive computer search. This approach is certainly feasible only when the covering radius is small and the optimal code consists of few codewords. We follow the approach in [31], where a search method for covering designs is discussed. We assume that for a given space and covering radius, M codewords are known to cover it, and we now want to prove that $M-1$ codewords cannot cover it. (Of course, it is possible that such a covering exists, and is then found in the search.) The algorithm is as follows.

1. Find the (lexicographically) first uncovered word. If no such word exists, a solution has been found.
2. If there are $M - 1$ codewords in the code, return.
3. Find all words that cover the uncovered word. For each such word
    (a) Add it to the code and mark the words that are covered by it.
    (b) Recursively call this algorithm.

(c) Delete the word and unmark the words that are covered by it.

4. Return.

Care must be taken in step 3(c), because words that are covered by the new codeword might also be covered by other codewords. This problem can be solved by indicating that a word is covered not by a flag but by an integer variable that shows how many times it is covered. The general algorithm can be speeded up by paying attention to the following facts [31]:

- We can backtrack whenever it is not possible (due to a sphere covering bound) to complete the covering with a total of at most $M-1$ codewords.
- If a word w is rejected on a level L, we need not consider this word until we backtrack from level L. This also holds for all words w' such that an equivalent code (at that moment) is obtained when w is replaced by w'. In particular, the first codeword can be fixed.

Our program has been checked against known exact values, to get an indication that it works properly. The program has been used to prove the following.

Theorem 19
$K_{3,2}(1,5;1) = 16$, $K_{3,2}(1,5;2) = 6$, $K_{3,2}(1,6;2) = 8$, $K_{3,2}(1,7;3) = 6$, $K_{3,2}(2,3;1) = 12$, $K_{3,2}(2,4;1) = 20$, $K_{3,2}(2,4;2) = 6$, $K_{3,2}(2,4;3) = 3$, $K_{3,2}(2,5;3) = 4$, $K_{3,2}(2,6;3) = 6$, $K_{3,2}(3,1;1) = 9$, $K_{3,2}(3,2;1) = 16$, $K_{3,2}(3,2;2) = 5$, $K_{3,2}(3,3;2) = 8$, $K_{3,2}(3,4;3) = 5$, $K_{3,2}(4,1;2) = 6$, $K_{3,2}(4,2;3) = 4$, $K_{3,2}(4,3;3) = 6$, $K_3(5,2) = 8$.

Note that for radius one, all upper bounds less than 36 are known to be exact. It is also interesting to see that a bound for a nonmixed ternary code has been fixed.

3 The Main Tables

In Table II upper bounds for $K_{3,2}(n,R)$ are presented. The lower bounds are not given, however, bounds that are known to be exact values are indicated by a period. In many cases there are several different constructions that can be used to prove a bound. Only one of these is then indicated, the one that in our opinion is the simplest. This implies that a reference to the original construction (and constructor) is only occasionally given; see [12] for an extensive list of such references. No references are given for the most trivial bounds (when there are less than 5 codewords in the code); codes attaining these can easily be constructed by hand.

Codes that are not known to have any (easily explained) structure are explicitly listed in Table III. Many of these have been found using simulated annealing; see [19, 27, 34] for information about how to use this method in searches for covering codes. The codes are given in a compressed notation, which is based on the idea of *skip-vectors* [2]. The procedure somewhat differs from that in [2], where a packing problem (constant weight codes) is discussed. The codewords are first sorted into lexicographic order. Then, going through all words in lexicographic order, the compressed notation shows the number of words between two consecutive codewords. The sequence $i, i, \ldots, i$ (k times) is abbreviated as i^k.

Example 6. The code $\{00, 21\}$ is a $(3, 2; 1, 1; 2)1$ code. The set of words in the space $F_3^1 F_2^1$ is $\{00, 01, 10, 11, 20, 21\}$ with the words lexicographically ordered, so the the code is expressed as $0, 4$ in compressed notation.

Codes that can be constructed by the matrix method are compressed using the same principles. These codes are given in Table IV. The columns of the **M**-matrix (which belong to the same space as the words of S) are also given by skip-vectors. They are separated from the words of S by a semi-colon.

Example 7. The binary Hamming code of length 3 can be given in matrix form with $S = \{00\} \in F_2^2$ and $\mathbf{M} = (1\ 1)^T$. In compressed notation this code is then given as $0; 3$.

In Table V acceptable partitions of codes that are used in partitioning constructions are given, except for that of the $(3, 11, 729)2$ Golay code and of a $(3, 6, 18)2$ code, which were given in Section 2.3 and [25], respectively. The digits in the strings indicate which codewords belong to the same subsets in a partition. The codewords can be found in Tables III and IV. For codes that are given in Table IV the codewords (the set W in Theorem 7) are first explicitly calculated. These codewords belong to a space $F_3^a F_2^b F_3^c F_2^d$. Whenever $b > 0$ and $c > 0$ these blocks with binary and ternary coordinates are swapped after which we have codewords that belong to $F_3^{a+c} F_2^{b+d}$. Finally these codewords are ordered lexicographically.

The notation {ij} in Table V indicates that the corresponding codeword belongs to several sets (C_i and C_j). From another point-of-view, this codeword occurs twice in the original code. It has been necessary to include some old codes in Tables III and IV to be able to give acceptable partitions for these.

Table II. $K_{3,2}(n_1, n_2; R)$, $n_1 + n_2 \leq 13$, $R \leq 3$.

n_1	n_2	$R=1$	$R=2$	$R=3$	n_1	n_2	$R=1$	$R=2$	$R=3$
0	1	1.	1.	1.	4	3	72^{c1}	18^{a1}	$6.^{a1}$
0	2	2.	1.	1.	4	4	132^{m}	24^{t10}	10^{a1}
0	3	2.	2.	1.	4	5	240^{m}	48^{c1}	16^{m}
0	4	4.	2.	2.	4	6	432^{m}	72^{b2}	24^{a1}
0	5	$7.^{r7}$	2.	2.	4	7	864^{c1}	144^{c1}	45^{n}
0	6	$12.^{r7}$	4.	2.	4	8	1296^{c5}	252^{b2}	68^{n}
0	7	$16.^{h}$	$7.^{a1}$	2.	4	9	2592^{c1}	480^{b2}	114^{n}
0	8	$32.^{c1}$	12^{a1}	4.	5	0	$27.^{c2}$	$8.^{n}$	3.
0	9	62^{n}	16^{a1}	$7.^{a1}$	5	1	54^{c1}	12^{n}	4.
0	10	120^{r24}	30^{n}	12^{a1}	5	2	96^{t16}	21^{n}	7^{n}
0	11	192^{r7}	44^{r15}	16^{a1}	5	3	168^{t16}	36^{c3}	12^{a1}
0	12	380^{r8}	78^{r27}	28^{r11}	5	4	324^{m}	64^{r26}	21^{a1}
0	13	750^{r12}	128^{a2}	42^{r11}	5	5	639^{m}	108^{c3}	32^{m}
1	0	1.	1.	1.	5	6	1206^{m}	192^{c4}	54^{t10}
1	1	2.	1.	1.	5	7	1944^{c3}	348^{e2}	90^{b2}
1	2	3.	2.	1.	5	8	3888^{c2}	672^{b3}	144^{t10}
1	3	$6.^{c1}$	2.	2.	6	0	73^{r19}	17^{n}	$6.^{c3}$
1	4	$8.^{c5}$	3.	2.	6	1	132^{t3}	27^{m}	9^{n}
1	5	$16.^{c1}$	$6.^{c1}$	2.	6	2	252^{c3}	48^{t16}	16^{n}
1	6	$24.^{c3}$	$8.^{a1}$	3.	6	3	468^{m}	72^{t16}	24^{t10}
1	7	48^{c1}	12^{m}	$6.^{c1}$	6	4	864^{m}	144^{c1}	44^{e4}
1	8	84^{m}	20^{n}	8^{a1}	6	5	1656^{m}	276^{b2}	72^{a1}
1	9	160^{m}	35^{n}	12^{a1}	6	6	2916^{c3}	519^{n}	96^{n}
1	10	288^{c3}	60^{m}	20^{a1}	6	7	5832^{c1}	960^{b2}	180^{b2}
1	11	548^{t16}	96^{t6}	32^{m}	7	0	186^{r19}	34^{n}	12^{c6}
1	12	1024^{c5}	179^{n}	52^{m}	7	1	333^{m}	54^{t9}	18^{t9}
2	0	3.	1.	1.	7	2	648^{m}	108^{c1}	33^{m}
2	1	4.	2.	1.	7	3	1296^{c1}	216^{c1}	54^{a1}
2	2	$6.^{n}$	3.	2.	7	4	2304^{t12}	396^{b2}	90^{t10}
2	3	$12.^{c1}$	4.	2.	7	5	4374^{c3}	720^{b2}	144^{c3}
2	4	$20.^{n}$	$6.^{a1}$	3.	7	6	8640^{m}	1296^{b2}	249^{n}
2	5	36^{c3}	11^{n}	4.	8	0	486^{r19}	81^{c3}	27^{c3}
2	6	64^{c4}	16^{m}	$6.^{a1}$	8	1	972^{c1}	162^{c1}	45^{m}
2	7	126^{c3}	28^{m}	11^{a1}	8	2	1728^{t12}	288^{b2}	72^{t12}
2	8	234^{m}	48^{t6}	16^{a1}	8	3	3456^{c1}	504^{b2}	108^{t10}
2	9	428^{m}	74^{m}	24^{m}	8	4	6480^{m}	972^{b2}	216^{c1}
2	10	768^{m}	144^{c3}	42^{n}	8	5	12960^{c1}	1620^{m}	324^{m}
2	11	1504^{m}	256^{c4}	70^{n}	9	0	1356^{t12}	219^{b2}	54^{b2}

Table II. (Continued)

n_1	n_2	$R=1$	$R=2$	$R=3$	n_1	n_2	$R=1$	$R=2$	$R=3$
3	0	$5.^{r16}$	3.	1.	9	1	2592^{c3}	396^{b2}	93^{b2}
3	1	$9.^{c3}$	3.	2.	9	2	4860^{m}	$729^{t2(2)}$	144^{n}
3	2	$16.^{n}$	$5.^{n}$	3.	9	3	9720^{c1}	1215^{m}	252^{b2}
3	3	$24.^{c4}$	$8.^{c4}$	3.	9	4	17496^{m}	$1944^{t2(2)}$	432^{t9}
3	4	48^{c1}	13^{n}	$5.^{a1}$	10	0	3645^{r1}	558^{b2}	108^{t10}
3	5	92^{m}	23^{t17}	8^{a1}	10	1	7047^{m}	$729^{t2(2)}$	189^{e3}
3	6	176^{m}	36^{m}	12^{m}	10	2	13122^{m}	1458^{c1}	324^{t9}
3	7	320^{m}	56^{n}	20^{n}	10	3	25192^{c4}	1944^{c4}	648^{c1}
3	8	576^{t12}	96^{m}	32^{m}	11	0	9447^{r28}	$729.^{h}$	243^{b2}
3	9	1120^{m}	192^{c1}	55^{n}	11	1	18894^{c1}	1458^{c1}	486^{c1}
3	10	2080^{t12}	358^{s2}	96^{a1}	11	2	37788^{c1}	2916^{c1}	729^{b1}
4	0	$9.^{h}$	3.	3.	12	0	27702^{r28}	2187^{c2}	729^{c2}
4	1	$18.^{c1}$	$6.^{c1}$	3.	12	1	52488^{m}	4374^{c1}	972^{b4}
4	2	36^{c1}	10^{s1}	4.	13	0	$59049.^{h}$	6561^{c2}	1215^{b5}

Key to Table II.

$ai =$ ADS of normal code and binary Hamming code of length $2^{i+1}-1$.
$bi =$ BDS of subnormal code and Code i (Section 2.3) punctured.
$ci =$ Corollary i.
$di =$ Direct sum of two codes.
$ei =$ Example i.
$h =$ Hamming or Golay code.
$m =$ Covering using matrix (Table IV).
$n =$ Explicitly listed code (Table III).
$ri =$ Reference [i].
$si =$ BDS of near-covering/almost subnormal code and Code i (Section 2.3) punctured. See Section 2.3.1.
$ti =$ Theorem i.

Table III. Explicitly listed codes.

$\mathbf{K_{3,2}(0,6;1) \leq 12}$

3, 9, 2, 5, 3, 2, 5, 0, 3, 5, 4, 9.

$\mathbf{K_{3,2}(0,9;1) \leq 62}$

0, 14, 3, 1, 3, 18, 4^2, 3, 12, 2, 17, 7, 4, 8, 12, 6, 4, 16, 8, 2, 9, 12^2, 3, 0, 13, 3, 18, 3, 2, 6, 3, 7, 13, 3, 0, 3^2, 17, 3, 11, 8, 4, 18, 6, 2, 8, 11, 9, 2, 19, 5, 4, 8, 2, 3, 18, 3, 13, 0, 3.

$\mathbf{K_{3,2}(0,10;2) \leq 30}$

62, 7, 50, 3, 26, 3, 6, 17, 79, 13, 22, 21, 142, 20, 13, 24, 17, 18, 3, 140, 17, 22, 17, 79, 13, 12, 30, 34, 21, 17.

$\mathbf{K_{3,2}(1,6;1) \leq 24}$

0, 12, 9, 2, 8, 10, 5, 4, 12, 4, 5, 10, 8, 2, 9, 12, 6^2, 3, 8^2, 4, 11, 10.

$\mathbf{K_{3,2}(1, 8; 2) \leq 20}$

9, 53, 52, 53, 21, 22, 58, 59, 20, 56, 8, 83, 12, 33, 44, 4, 48, 37, 59, 8.

$\mathbf{K_{3,2}(1, 9; 2) \leq 35}$

26, 17, 24, 91, 85, 17, 43, 52, 35, 67, 5, 58, 17, 9, 100, 51, 53, 93, 6^2, 28, 48, 24, 116, 41, 8, 35, 4, 40, 52, 53, 40, 91, 40, 8.

$\mathbf{K_{3,2}(1, 12; 2) \leq 179}$

50, 42, 15, 33, 32, 17, 59, 517, 26, 54, 2, 22, 53, 2, 77, 272, 44, 4, 20, 117, 4, 21, 26, 5, 45, 51, 8, 38, 8, 51, 44, 515, 45, 51, 8, 38, 8, 51, 44, 10, 44, 4, 20, 117, 4, 21, 26, 264, 26, 54, 2, 22, 53, 2, 77, 522, 53, 12, 73, 32, 74, 2, 14, 0, 1, 47, 62, 0, 62, 47, 1, 0, 536, 39, 8, 134, 8, 43, 270, 94, 2, 44, 2, 90, 43, 52, 4, 74, 4, 52, 576, 52, 4, 74, 4, 52, 39, 94, 2, 44, 2, 90, 278, 39, 8, 134, 8, 43, 547, 26, 14, 10, 27, 12, 10, 12, 27, 10, 14, 26, 30, 32, 25, 11, 13, 12, 21, 10, 21, 12, 13, 37, 20, 627, 4, 21, 20, 4, 84, 288, 66, 8, 70, 8, 18, 78, 39, 2, 140, 2, 40, 539, 39, 2, 140, 2, 40, 29, 66, 8, 70, 8, 18, 421, 4, 21, 20, 4, 84, 535, 3, 1, 18, 31, 62, 95, 17, 2, 3.

$\mathbf{K_{3,2}(2, 2; 1) \leq 6}$

3, 4, 3, 10, 5, 0.

$\mathbf{K_{3,2}(2, 4; 1) \leq 20}$

5, 9, 6, 1, 10, 4, 7, 9, 11, 6, 5, 9, 4^2, 1, 2, 4, 9, 8, 9.

$\mathbf{K_{3,2}(2, 5; 2) \leq 11}$

14, 2, 27, 36^2, 32, 10, 36, 27, 54, 0.

$\mathbf{K_{3,2}(2, 7; 2) \leq 30}$

0, 82, 24, 47, 6, 91, 0, 82, 24, 73, 2, 34, 6, 47, 24^2, 4, 107, 2, 34, 6, 43, 37, 4, 139, 2, 38, 30, 37, 4.

$\mathbf{K_{3,2}(2, 10; 3) \leq 42}$

68, 366, 99, 456, 205, 42, 48, 114, 807, 172, 280, 44, 163, 146, 104, 146, 163, 44, 345, 10, 285, 456, 57, 78, 81, 163, 44, 1781, 114, 48, 42, 420, 10, 419, 172, 264, 44, 163, 81, 78, 118, 366.

$\mathbf{K_{3,2}(2, 11; 3) \leq 70}$

0, 62, 391, 48, 405, 34, 289, 24, 364, 12, 181, 6, 220, 62, 651, 40, 88, 36, 482, 20, 43, 18, 240, 10, 2394, 62, 651, 40, 88, 36, 482, 20, 43, 18, 240, 10, 346, 62, 391, 48, 405, 34, 289, 24, 364, 12, 181, 6, 6584, 6, 118, 10, 51, 12, 175, 18, 43, 20, 104, 24, 289, 34, 27, 36, 88, 40, 210, 48, 391, 62.

$\mathbf{K_{3,2}(3, 2; 1) \leq 16}$

0, 2, 9, 0, 16, 0, 9, 3, 5, 6, 1, 15, 3, 6^2, 2.

$\mathbf{K_{3,2}(3, 2; 2) \leq 5}$

1^2, 54, 0, 40.

$\mathbf{K_{3,2}(3, 4; 2) \leq 13}$

17, 33, 5, 44, 5, 67, 4, 40, 44, 62, 4, 20, 44.

$\mathbf{K_{3,2}(3, 7; 2) \leq 56}$

3, 0, 43, 6, 16, 6, 43, 0, 540, 4, 14, 36, 2, 11, 4, 182, 12, 38, 4, 14, 439, 12, 38, 4, 14, 154, 6, 43, 0, 22, 0, 43, 6, 154, 14, 4, 38, 12, 439, 14, 4, 38, 12, 186, 12,

38, 4, 14, 541, 0, 43, 6, 16, 6, 43, 0.
$\mathbf{K_{3,2}(3,7;2) \leq 60}$
26, 2, 11, 4^2, 0, 22, 0, 4^2, 11, 2, 538, 6, 112, 6, 128, 6, 112, 6, 384, 6, 112, 6, 153, 4, 12, 0, 5, 2, 20, 2, 5, 0, 12, 4, 153, 6, 112, 6, 384, 6, 112, 6, 128, 6, 112, 6, 539, 0, 13, 2, 3, 4, 18, 4, 3, 2, 13, 0.
$\mathbf{K_{3,2}(3,7;3) \leq 20}$
15, 12, 6, 47, 24, 3, 1551, 52, 4, 11, 2, 53, 1555, 24, 9, 2, 11, 4, 17, 6.
$\mathbf{K_{3,2}(3,9;3) \leq 55}$
3072, 91, 13, 103, 89, 103, 13, 91, 54, 87, 45, 43, 49, 43, 45, 87, 749, 43, 139, 87, 158, 103, 207, 91, 280, 289, 87, 139, 43, 186, 91, 140, 66, 103, 1685, 155, 298, 1039, 50, 40, 46, 139, 20, 22, 80, 90, 1049, 92, 78, 24, 18, 139, 48, 38, 52.
$\mathbf{K_{3,2}(4,1;1) \leq 18}$
3, 0, 26, 0, 9, 4, 18, 4, 9, 0, 8, 0, 22, 0, 9, 4, 27, 0.
$\mathbf{K_{3,2}(4,7;3) \leq 45}$
113, 32, 17, 35, 40, 160, 17, 35, 40, 15, 12, 112, 541, 13, 27, 54, 2206, 13, 27, 54, 1026, 12, 97, 454, 0, 27, 6^2, 10, 5011, 12, 3, 10, 6^2, 11, 0, 14, 0, 11, 6^2, 10, 3, 12.
$\mathbf{K_{3,2}(4,8;3) \leq 68}$
256, 62, 143, 35, 8, 262, 8, 35, 143, 62, 1365, 20, 47, 11, 2, 4272, 11, 2, 44, 2, 11, 4, 171, 2, 11, 47, 20, 3413, 62, 143, 35, 8, 274, 35, 8, 134, 8, 35, 14, 1365, 4, 11, 2, 44, 2, 11, 3247, 20, 47, 11, 2, 1196, 2, 11, 47, 20, 1624, 8, 35, 143, 62, 768, 14, 35, 8, 134, 8, 35.
$\mathbf{K_{3,2}(4,9;3) \leq 114}$
563, 19, 67, 112, 34, 72, 59, 43, 75, 485, 45, 19, 67, 118, 28, 78, 59, 43, 2649, 43, 59, 76, 30, 116, 67, 19, 1586, 492, 1540, 1017, 561, 3086, 43, 59, 74, 32, 114, 67, 19, 96, 43, 59, 74, 32, 114, 67, 19, 34, 18, 6703, 19, 67, 112, 34, 72, 59, 43, 607, 19, 67, 118, 28, 78, 59, 43, 25, 1029, 1576, 16, 43, 59, 76, 30, 116, 67, 19, 3123, 2032, 1081, 43, 59, 76, 30, 116, 67, 19, 31, 2111, 43, 59, 74, 32, 114, 67, 19, 593, 2576, 19, 67, 118, 28, 78, 59, 43, 1564, 16, 49, 19, 67, 112, 34, 72, 59, 43.
$\mathbf{K_{3,2}(5,0;2) \leq 8}$
6, 46, 10, 16, 69, 51, 7, 27.
$\mathbf{K_{3,2}(5,1;2) \leq 12}$
1, 104, 26, 6, 60, 6, 26, 50, 71, 27, 0, 81.
$\mathbf{K_{3,2}(5,2;1) \leq 96}$
1^2, 14, 13, 7, 9, 1, 14^2, 10, 2, 9, 5, 7, 5, 16, 2, 3, 2^2, 38, 13, 16, 6, 8, 5, 26, 1, 10, 6, 0, 17, 15, 5, 3, 6, 12, 5, 7, 18, 7, 21, 10, 1, 9, 10^2, 16, 1, 14, 12, 8, 9, 1, 12, 21, 7, 16, 7, 5, 14, 4, 3, 5, 15, 17, 2, 4, 12, 1, 24, 5, 10, 4, 18, 13, 36, 4, 0, 3, 4, 14, 5, 7, 5, 9, 4, 8, 16, 12, 1, 9, 7, 13, 16, 1.
$\mathbf{K_{3,2}(5,2;2) \leq 21}$
31, 15, 36, 46, 4, 15, 18, 15, 156, 39, 15, 10, 51, 15, 36, 348, 0, 50, 0, 50, 0.
$\mathbf{K_{3,2}(5,2;3) \leq 7}$
190, 10, 11, 363, 24, 61, 2.

$\mathbf{K_{3,2}(5,3;1) \leq 168}$

1, 0^5, 32, 24, 15, 22, 0, 26, 1, 0, 32, 17, 0, 1, 3, 22, 8, 15, 14, 27, 1, 0, 8, 7, 1, 0, 1, 3, 74, 1, 0, 26, 0, 1, 34, 9, 0, 1, 18, 8, 50, 1, 0, 2, 0, 1, 19, 10, 1, 0, 1, 38, 31, 8, 7, 10, 1, 0, 25, 14, 15, 33, 0, 1, 12, 0, 1, 38, 1, 0, 24, 0, 22, 17, 0, 1, 19, 32, 0^5, 32, 19, 1, 0, 17, 22, 0, 24, 0, 1, 38, 1, 0, 12, 1, 0, 33, 15, 14, 25, 0, 1, 10, 7, 8, 31, 38, 1, 0, 1, 10, 19, 1, 0, 2, 0, 1, 50, 8, 18, 1, 0, 9, 34, 1, 0, 26, 0, 1, 74, 3, 1, 0, 1, 7, 8, 0, 1, 27, 14, 15, 8, 22, 3, 1, 0, 17, 32, 0, 1, 26, 0, 22, 15, 24, 32, 0^5.

$\mathbf{K_{3,2}(5,4;1) \leq 336}$

0^3, 2, 0^2, 1, 0^5, 52, 11, 52, 3, 27, 3, 26, 11, 4, 3, 40, 0, 4, 3, 1, 0, 52, 11, 32, 4, 0, 2, 0, 1, 7, 3, 26, 11, 20, 3, 27, 3, 10, 11, 49, 0, 4, 3, 1, 0, 4, 11, 3, 11, 0, 4, 0, 2, 0, 1, 7, 3, 136, 0, 4, 3, 1, 0, 49, 4, 0, 2, 0, 1, 54, 11, 16, 4, 0, 2, 0, 1, 22, 11, 20, 3, 88, 0, 4, 3, 1, 0, 1, 4, 0, 2, 0, 1, 39, 3, 8, 0, 4, 3, 1, 0, 5, 3, 58, 11, 51, 11, 20, 3, 11, 3, 8, 0, 4, 3, 1, 0, 53, 3, 10, 11, 19, 11, 64, 4, 0, 2, 0, 1, 19, 4, 0, 2, 0, 1, 68, 0, 4, 3, 1, 0, 36, 11, 4, 3, 26, 11, 32, 4, 0, 2, 0, 1, 39, 3, 55, 0^2, 2, 0^2, 1, 0^5, 52, 11, 33, 0, 4, 3, 1, 0, 37, 3, 26, 11, 4, 3, 39, 4, 0, 2, 0, 1, 68, 0, 4, 3, 1, 0, 18, 0, 4, 3, 1, 0, 69, 3, 27, 3, 10, 11, 48, 4, 0, 2, 0, 1, 6, 11, 3, 11, 20, 3, 59, 3, 58, 11, 0, 4, 0, 2, 0, 1, 6, 11, 33, 0, 4, 3, 1, 0, 1, 4, 0, 2, 0, 1, 86, 11, 20, 3, 24, 0, 4, 3, 1, 0, 21, 3, 56, 0, 4, 3, 1, 0, 49, 4, 0, 2, 0, 1, 134, 11, 1, 0, 4, 3, 1, 0, 5, 3, 11, 3, 7, 4, 0, 2, 0, 1, 55, 3, 10, 11, 19, 11, 20, 3, 26, 11, 1, 0, 4, 3, 1, 0, 37, 3, 55, 4, 0, 2, 0, 1, 38, 11, 4, 3, 26, 11, 19, 11, 52, 3, 55, 0^2, 2, 0^2, 1, 0^5.

$\mathbf{K_{3,2}(6,0;1) \leq 73}$

3, 12, 9, 17, 1, 2^2, 3, 42, 7^2, 4, 20, 3, 12, 10, 3, 2, 12, 7, 22, 10, 8, 11, 16, 5, 3, 24, 1, 11, 7, 1, 8^2, 5, 3, 18, 11, 7, 15, 7, 4, 20, 2, 3, 9, 15, 0, 7, 24, 6, 32, 0^2, 3, 9, 3, 9, 13, 1, 6, 8, 9, 21, 12, 9, 0, 14, 5, 4, 1, 3, 7.

$\mathbf{K_{3,2}(6,0;2) \leq 17}$

48, 40, 56, 27, 12, 29, 68, 37, 65, 12, 21, 38, 34, 14, 49, 51, 73.

$\mathbf{K_{3,2}(6,1;1) \leq 132}$

8^2, 3, 2^2, 12, 8, 2, 4, 3, 0, 9, 1, 4, 32, 7, 4, 19, 3, 6, 7, 4, 1, 9, 0, 3, 29, 2^2, 12, 14, 12, 2^2, 29, 3, 0, 9, 1, 4, 7, 6, 3, 19, 4, 7, 32, 4, 1, 9, 0, 3, 4, 2, 8, 12, 2^2, 3, 8^2, 0, 30, 14, 22, 6, 18, 8, 11, 18, 2, 6, 25, 24, 3, 8, 18, 12, 0, 12, 18, 8, 3, 24, 25, 6, 2, 18, 11, 8, 18, 6, 22, 14, 30, 0^3, 30, 14, 22, 6, 18, 8, 11, 18, 2, 6, 25, 24, 3, 8, 18, 12, 0, 12, 18, 8, 3, 24, 25, 6, 2, 18, 11, 8, 18, 6, 22, 14, 30, 0.

$\mathbf{K_{3,2}(6,1;2) \leq 31}$

47, 23, 54, 1, 68, 6, 1, 21, 1, 26, 23, 234, 1, 57, 1, 21, 1, 14, 77, 23, 54, 1, 520, 1, 0, 74, 1, 0, 74, 1, 0.

$\mathbf{K_{3,2}(6,1;3) \leq 9}$

207, 60, 203, 303, 62, 56, 81, 48, 65.

$\mathbf{K_{3,2}(6,2;3) \leq 16}$

20, 39, 18, 461, 268, 431, 192, 15, 36, 132, 432, 266, 590, 0^3.

$\mathbf{K_{3,2}(6,3;2) \leq 78}$

0, 42, 0, 11, 6, 24, 6, 13^2, 0, 30, 0, 11, 6, 38, 292, 573, 4, 99, 2, 100, 2, 220, 2, 99, 4, 98, 4, 651, 2, 99, 4, 98, 4, 5, 251, 6, 11, 0, 30, 0, 27, 6, 24, 6, 11, 0, 260, 4, 99, 2, 100, 2, 651, 4, 99, 2, 100, 2, 220, 2, 99, 4, 98, 4, 440, 464, 6, 11, 0, 30, 0, 27, 6, 24, 6, 11, 0.

$\mathbf{K_{3,2}(6,6;2) \le 519}$

13, 8, 4, 11, 2, 6, 206, 59, 451, 59, 209, 10, 11, 12, 205, 59, 146, 35, 140, 59, 259, 59, 216, 4, 11, 2, 22, 59, 3, 59, 18, 35, 609, 4, 11, 2, 229, 35, 204, 59, 90, 19, 84, 59, 465, 10, 11, 12, 36, 19, 186, 46, 4, 11, 2, 278, 59, 195, 59, 169, 46, 4, 11, 2, 278, 59, 67, 59, 282, 19, 107, 19, 290, 10, 11, 12, 141, 59, 131, 59, 146, 35, 409, 8, 4, 11, 2, 6, 206, 59, 145, 10, 11, 12, 269, 59, 210, 35, 289, 4, 11, 2, 107, 4, 11, 2, 470, 59, 210, 35, 12, 59, 3, 59, 579, 59, 18, 35, 12, 59, 152, 4, 11, 2, 37, 35, 332, 59, 216, 4, 11, 2, 356, 10, 11, 12, 228, 19, 532, 59, 24, 4, 11, 2, 135, 37, 19, 276, 59, 67, 59, 195, 59, 17, 10, 11, 12, 412, 35, 140, 59, 131, 59, 67, 59, 464, 8, 4, 11, 2, 6, 28, 6, 2, 11, 4, 8, 464, 59, 67, 59, 131, 59, 148, 19, 429, 2, 11, 4, 24, 59, 195, 59, 67, 59, 268, 35, 110, 3, 50, 2, 11, 4, 24, 59, 524, 35, 229, 2, 11, 4, 354, 12, 11, 10, 209, 59, 340, 19, 36, 12, 11, 10, 145, 20, 38, 12, 35, 18, 59, 579, 59, 3, 20, 38, 12, 35, 210, 59, 461, 12, 11, 10, 91, 12, 11, 10, 290, 19, 218, 59, 278, 2, 11, 4, 152, 59, 206, 6, 2, 11, 4, 8, 417, 19, 154, 59, 131, 59, 150, 2, 11, 4, 289, 35, 91, 35, 274, 59, 67, 59, 278, 2, 11, 4, 107, 18, 89, 59, 195, 59, 223, 3, 50, 2, 11, 4, 225, 35, 37, 2, 11, 4, 472, 59, 76, 35, 82, 59, 212, 19, 228, 12, 11, 10, 602, 35, 18, 59, 3, 8, 50, 13, 12, 11, 10, 209, 59, 259, 59, 148, 19, 154, 59, 214, 2, 11, 4, 216, 59, 451, 59, 206, 6, 2, 11, 4, 8, 26, 8, 4, 11, 2, 6, 206, 59, 451, 59, 209, 10, 11, 12, 205, 59, 146, 35, 140, 59, 259, 59, 216, 4, 11, 2, 22, 59, 3, 59, 18, 35, 609, 4, 11, 2, 229, 35, 204, 59, 90, 19, 84, 59, 465, 10, 11, 12, 36, 19, 186, 46, 4, 11, 2, 278, 59, 195, 59, 169, 46, 4, 11, 2, 278, 59, 67, 59, 282, 19, 107, 19, 290, 10, 11, 12, 141, 59, 131, 59, 146, 35, 409, 8, 4, 11, 2, 6, 206, 59, 145, 10, 11, 12, 269, 59, 210, 35, 289, 4, 11, 2, 107, 4, 11, 2, 470, 59, 210, 35, 12, 59, 3, 59, 579, 59, 18, 35, 12, 59, 152, 4, 11, 2, 37, 35, 332, 59, 216, 4, 11, 2, 356, 10, 11, 12, 228, 19, 532, 59, 24, 4, 11, 2, 135, 37, 19, 276, 59, 67, 59, 195, 59, 17, 10, 11, 12, 412, 35, 140, 59, 131, 59, 67, 59, 464, 8, 4, 11, 2, 6.

$\mathbf{K_{3,2}(6,6;3) \le 96}$

323, 8, 35, 79, 62, 207, 35, 8, 198, 8, 35, 210, 8, 35, 94, 35, 8, 7257, 2, 11, 815, 20, 815, 11, 2, 1771, 20, 815, 11, 2, 812, 2, 11, 5236, 11, 2, 812, 2, 11, 815, 20, 2069, 62, 79, 35, 8, 198, 8, 35, 207, 62, 192, 62, 67, 8, 35, 2085, 2, 11, 815, 20, 815, 11, 2, 5228, 2, 11, 815, 20, 815, 11, 2, 1776, 11, 2, 812, 2, 11, 815, 20, 7268, 35, 8, 70, 8, 35, 207, 62, 207, 35, 8, 210, 35, 8, 67, 62.

$\mathbf{K_{3,2}(7,0;2) \le 34}$

153, 14, 3, 12, 86, 9, 12, 119, 209, 12, 27, 122, 3, 12, 129, 112, 5, 9, 12, 208, 120, 12, 54, 93, 12, 231, 103, 9, 12, 5, 12, 96, 3, 12.

$\mathbf{K_{3,2}(7,1;2) \le 54}$

75, 68, 27, 140, 27, 68, 170, 14, 75, 92, 75, 74, 110, 74, 75, 92, 75, 14, 159, 60, 57, 118, 57, 60, 142, 60, 63, 118, 3, 60, 196, 60, 3, 118, 63, 60, 105, 20, 171, 44, 171, 20, 41, 32, 183, 2, 216, 26, 14, 26, 216, 2, 183, 32.

$\mathbf{K_{3,2}(7, 6; 3) \leq 249}$
459, 0, 43, 210, 0, 43, 594, 0, 43, 658, 0, 43, 594, 0, 43, 210, 0, 43, 455, 6, 43, 0, 779, 6, 43, 0, 779, 6, 43, 0, 21130, 72, 255, 310, 255, 72, 703, 182, 255, 200, 255, 182, 323, 0, 43, 786, 0, 43, 786, 0, 43, 5541, 11, 2, 107, 255, 260, 11, 2, 240, 11, 2, 107, 703, 132, 11, 2, 240, 11, 2, 235, 255, 132, 11, 2, 372, 2, 11, 4, 811, 2, 11, 4, 811, 2, 11, 4, 15910, 11, 4, 107, 255, 258, 11, 4, 238, 11, 4, 107, 703, 130, 11, 4, 238, 11, 4, 235, 255, 130, 11, 4, 370, 4, 11, 2, 811, 4, 11, 2, 811, 4, 11, 2, 5537, 171, 0, 254, 0, 210, 255, 171, 0, 702, 0, 82, 255, 299, 0, 254, 0, 82, 368, 54, 776, 54, 776, 54, 5575, 72, 255, 310, 255, 72, 703, 182, 255, 200, 255, 182, 323, 0, 43, 786, 0, 43, 786, 0, 43, 16023, 54, 200, 54, 584, 54, 648, 54, 584, 54, 200, 54, 451, 0, 43, 786, 0, 43, 786, 0, 43, 5536, 4, 14, 124, 255, 238, 4, 14, 235, 4, 14, 124, 703, 110, 4, 14, 235, 4, 14, 252, 255, 110, 4, 14, 372, 2, 11, 4, 811, 2, 11, 4, 811, 2, 11, 4, 21133, 82, 43, 211, 43, 256, 255, 82, 43, 659, 43, 128, 255, 210, 43, 211, 43, 128, 323, 54, 776, 54, 776, 54.

$\mathbf{K_{3,2}(8, 0; 2) \leq 81}$
4, 48, 9, 35, 21^2, 35, 9, 48, 983, 39, 18, 35, 21, 12, 35, 18, 48, 245, 48, 18, 35, 12, 21, 35, 18, 39, 737, 48, 18, 35, 12, 21, 35, 18, 39, 254, 48, 9, 35, 21^2, 35, 9, 48, 254, 39, 18, 35, 21, 12, 35, 18, 48, 737, 39, 18, 35, 21, 12, 35, 18, 48, 245, 48, 18, 35, 12, 21, 35, 18, 39, 983, 48, 9, 35, 21^2, 35, 9, 48.

$\mathbf{K_{3,2}(9, 2; 3) \leq 144}$
22^2, 16, 15, 6, 457, 100, 163, 54, 324, 54, 189, 16, 39, 132, 100, 330, 100, 111, 54, 505, 8, 30, 15, 11723, 16, 39, 1338, 8, 30, 15, 1348, 22, 16, 15, 6, 3530, 54, 108, 54, 379, 100, 379, 54, 376, 54, 111, 100, 9883, 54, 108, 54, 379, 100, 379, 54, 376, 54, 111, 100, 3480, 8, 30, 15, 462, 54, 163, 100, 379, 54, 130, 22, 16, 15, 6, 130, 54, 376, 54, 212, 54, 461, 16, 39, 2966, 16, 39, 1338, 8, 30, 15, 1348, 22, 16, 15, 6, 8799, 16, 39, 1338, 8, 30, 15, 1348, 22, 16, 15, 6, 3530, 54, 108, 54, 379, 100, 379, 54, 376, 54, 111, 100, 12229, 22, 16, 15, 6, 457, 100, 163, 54, 324, 54, 189, 16, 39, 132, 100, 330, 100, 111, 54, 505, 8, 30, 15.

$\mathbf{K_{4,3,2}(1, 1, 6; 2) \leq 18}$
14, 4, 12, 28, 132, 41, 4, 45, 165, 37, 10, 33, 14, 112, 14, 33, 10, 37.

$\mathbf{K_{4,3,2}(1, 2, 3; 2) \leq 8}$
41, 20, 53, 14, 54, 18, 10, 6.

$\mathbf{K_{5,3}(1, 6; 1) \leq 252}$
5, 9, 3, 42, 0, 12, 32, 12, 9, 2, 12^2, 3, 12, 3, 13, 3, 9, 62, 12, 3, 13, 3, 9, 13, 0, 12, 9^2, 3, 34, 12^2, 28, 12, 9, 3, 12, 9, 2, 12^2, 34, 3, 9, 35, 12, 3, 41, 0, 12, 9^2, 3, 14, 3, 9, 11, 3, 12, 32, 9, 12, 33, 9, 3, 34, 12^2, 4, 12, 0, 12^2, 0, 35, 12^2, 34, 3, 9, 11, 3, 12, 8, 9, 3, 9^2, 12, 60, 9, 3, 9^2, 12, 1, 12^2, 4, 12, 0, 41, 3, 12, 37, 3, 9, 5, 25, 4, 13, 22, 1, 16, 3, 24, 11, 27, 19, 7, 19, 13, 22, 3, 9, 22, 15, 21, 27, 3, 2, 12, 17, 12, 16, 29, 12, 6, 5, 33, 5, 15, 21, 16, 8, 33, 12, 2, 7, 34, 0, 9, 1, 36, 12, 0, 12, 30, 7, 4, 16, 23, 5, 12^2, 11, 20, 22, 7, 31, 1, 37, 1, 2, 12, 27, 14, 12, 23, 12, 2, 15, 19, 21, 11, 16, 13, 12, 0, 38, 9, 5, 35, 9, 3, 34, 9, 6, 7, 30, 3, 6, 25, 13, 12, 2, 9, 15, 22, 6, 13, 11, 12, 21, 23, 0, 36, 13, 16, 9, 3, 25, 12, 9, 22, 12^2, 25, 12^2, 22, 9, 12, 25, 3, 9, 16, 13, 36, 0, 23, 21, 12, 11, 13, 6, 22, 15, 9, 2, 12.

Table IV. Coverings using matrices.

$\mathbf{K_{3,2}(0,10;2) \le 118}$ (0 ternary coordinates, 9 binary coordinates in H)
384; 2, 0, 17, 2, 6, 7, 1, 8, 18, 3, 2, 5, 17, 9, 4, 8, 4, 12, 4^2, 14, 2, 9, 11, 2, 25, 15, 8, 2, 0, 4^2, 6, 12, 14, 9, 11, 0, 16, 4, 2, 3, 5, 0, 5, 36, 8, 1, 7, 6, 4, 14, 10, 3, 6, 12^2, 4^2.

$\mathbf{K_{3,2}(1,7;2) \le 12}$ (1,6)
62; 33, 20, 24, 4, 70, 32.

$\mathbf{K_{3,2}(1,8;1) \le 84}$ (1,7)
127; 13, 8, 4, 7, 0, 19, 9, 5^2, 2, 19, 4, 9, 10, 7, 2, 5, 24, 11, 8, 5, 6, 14, 2^2, 5^2, 6, 13, 26, 0, 4, 12, 2, 21, 5, 4, 3, 13, 14, 4, 1.

$\mathbf{K_{3,2}(1,9;1) \le 160}$ (1,7)
30, 89; 6^2, 7, 4, 8, 4, 7, 14, 1, 8^3, 7, 20, 9, 8, 7, 10, 6, 2, 7, 8, 10, 18, 7, 13, 8, 5, 2, 24, 11, 22, 14, 3, 2, 3, 20, 3, 2, 3.

$\mathbf{K_{3,2}(1,10;2) \le 60}$ (1,9)
511; 21, 74^2, 50, 79, 26, 41, 14, 53, 44, 33, 64, 112, 60, 69, 30, 5, 46, 25, 34, 71, 25, 42, 40, 96, 33, 90, 86, 20, 41.

$\mathbf{K_{3,2}(1,11;3) \le 32}$ (1,9)
460, 35; 415, 40, 66, 80, 585, 75, 62, 51.

$\mathbf{K_{3,2}(1,12;3) \le 52}$ (1,10)
31, 960; 45, 294, 238, 215, 78, 539, 112, 448, 16, 229, 46, 434, 78.

$\mathbf{K_{3,2}(2,9;3) \le 24}$ (2,8)
254; 88, 80, 36, 38, 840, 36, 160, 142, 369, 140, 172, 34.

$\mathbf{K_{3,2}(2,6;2) \le 16}$ (2,4)
13, 0; 34, 6, 20, 54.

$\mathbf{K_{3,2}(2,7;2) \le 28}$ (2,6)
56; 133, 24, 12, 4, 31, 38, 10, 22, 18, 85, 22, 18, 51, 22.

$\mathbf{K_{3,2}(2,8;1) \le 234}$ (2,7)
96; 15, 1^2, 12, 4, 19, 2, 13, 11, 15, 6, 12, 13, 20, 0, 12, 11^2, 3, 4, 3, 10, 5, 0, 17, 15, 1, 22, 16, 3, 22, 11, 5^2, 1, 20, 1, 4, 2, 6, 20, 15, 3, 14, 2, 19, 4, 1, 11, 9, 1, 13, 1, 19, 0, 1, 14, 3, 24, 11, 9, 1, 22, 11, 0, 9, 18, 3, 0, 11, 10, 15, 3, 24, 12, 0, 9, 7^2, 3, 10, 19, 3^2, 7, 1, 0, 8, 15, 4, 7, 11, 4, 16, 10, 6, 1, 17, 1, 14, 11, 24, 0, 3, 11, 10, 0, 11, 21, 3, 14, 1, 25, 1, 6, 16, 2.

$\mathbf{K_{3,2}(2,9;1) \le 428}$ (2,7)
31, 64; 4, 12, 8, 14, 4, 13, 10^2, 12, 0, 10, 9, 13, 18, 6, 18, 6, 3, 5, 8, 1, 25, 6, 10, 15, 6, 18, 2, 46, 4, 25, 10, 3, 13, 6, 12, 10^2, 16, 9, 8, 2, 10, 12, 11, 3, 4, 20, 4, 20, 8, 12, 4, 25, 1, 10, 27, 0, 20, 4, 9, 12, 25, 2, 34, 1, 12, 7, 0, 13, 15, 4, 16^2, 5^2, 4^2, 7, 18, 6^2, 19, 0, 16, 4, 5, 12, 9, 2, 12, 4, 11, 2, 21, 0, 11, 6, 3, 10, 8, 2, 9, 4, 2, 3, 12.

$\mathbf{K_{3,2}(2,9;2) \le 74}$ (2,8)
224; 13, 8, 146, 8, 187, 6, 45, 20, 266, 0, 19, 6, 39, 20, 173, 6, 59, 104, 59, 16, 20, 144, 8, 49, 26, 160, 6, 19, 0, 131, 26, 177, 8, 51, 6, 156, 6.

$\mathbf{K_{3,2}(2,9;3) \leq 24}$ (2,8)
254; 88, 80, 36, 38, 840, 36, 160, 142, 369, 140, 172, 34.
$\mathbf{K_{3,2}(2,10;1) \leq 768}$ (2,5)
14, 3, 1, 3, 5; 6, 5, 10, 9, 8, 37, 10, 1, 2, 16, 27, 8, 4, 0, 10, 2, 32, 14, 2, 9, 8, 16, 10, 9.
$\mathbf{K_{3,2}(2,11;1) \leq 1504}$ (2,7)
83, 16, 3, 7; 63, 6, 4^2, 0, 11, 3, 10^2, 2, 18, 8, 1, 2, 8, 2^2, 6, 78, 0, 11, 35, 10, 9, 4, 0, 23, 10, 7, 2, 6, 13, 4, 5, 11, 1, 10, 17, 2, 38, 4^2, 8, 18, 19^2, 0, 11, 8^2, 2, 0, 2, 18, 35, 11, 30, 10, 14, 3, 8, 12, 18, 10, 1, 11, 5, 4, 14, 17, 12, 3, 32, 10, 35, 17, 2, 1, 8, 26, 0, 6, 2^2, 4, 13, 1, 6, 1, 11, 5, 6, 9, 15.
$\mathbf{K_{3,2}(3,5;1) \leq 92}$ (3,4)
12; 0, 21, 6, 13, 11, 7, 8^2, 0, 22, 0, 8, 16, 7, 6, 18, 6, 10, 11, 7, 14, 5, 0, 22, 0, 5, 22, 7, 5, 8, 4^2, 12, 8, 7, 2, 11, 8, 1, 2, 12, 10, 9, 8, 11, 2.
$\mathbf{K_{3,2}(3,6;1) \leq 176}$ (3,4)
3, 8; 5, 2, 9, 5, 9, 12, 2, 12, 5, 9, 2, 5, 8, 4, 11, 2, 18, 6, 3, 25, 6, 5, 14, 6, 22, 6, 7, 5, 9, 4, 5, 28, 8^3, 10, 21, 8, 9, 3, 9, 5, 10, 9.
$\mathbf{K_{3,2}(3,6;2) \leq 36}$ (2,5)
31, 96; 34, 2, 66, 18, 146, 5.
$\mathbf{K_{3,2}(3,6;3) \leq 12}$ (3,5)
31; 74, 130, 261, 74, 81, 53.
$\mathbf{K_{3,2}(3,7;1) \leq 320}$ (3,4)
7, 2, 1; 15, 9, 6, 5, 0, 10, 9, 10, 17, 13, 12, 5, 20, 1, 16, 9, 18, 7, 9, 18, 5, 13, 0, 4, 8, 9, 18, 5, 12, 17, 7, 22, 5, 0, 7, 9, 3, 11, 16, 9.
$\mathbf{K_{3,2}(3,8;2) \leq 96}$ (3,6)
40, 7; 7, 16, 11, 220, 60, 220, 6, 26, 131, 60, 30, 4, 24, 200, 19, 14, 179, 6, 26, 201, 16, 11, 28, 60.
$\mathbf{K_{3,2}(3,8;3) \leq 32}$ (3,6)
52, 3; 361, 12, 205, 22, 14, 10, 977, 16.
$\mathbf{K_{3,2}(3,9;1) \leq 1120}$ (3,5)
19, 1, 0, 1; 10^2, 11, 28, 1, 6, 0, 15, 19, 6, 19, 2, 13, 16, 12, 23, 0^2, 19, 22, 6, 26, 6, 4, 12, 4, 19, 0, 17, 26, 4, 26, 2, 20, 4, 2, 17, 0, 22, 16, 17, 10, 11, 10, 6, 10, 16, 0, 17, 9, 6, 9, 11, 12, 10, 28, 8, 16, 11, 5, 10, 5^2, 4, 1, 0, 31, 12, 24, 0.
$\mathbf{K_{3,2}(3,10;1) \leq 2144}$ (3,5)
14, 7, 2, 0, 1; 19, 4, 13, 4, 33, 6, 23, 18, 0^2, 37, 2, 7, 31, 2, 23, 16, 1, 24, 6^2, 10, 25, 8, 20, 13, 6, 20, 2, 1, 24, 0, 10, 1, 12, 24, 3, 2, 19, 22, 5, 7, 4, 8, 24, 16, 7, 18, 7, 16, 11^2, 0, 26, 13, 0, 31, 2, 7, 0, 5, 13, 20, 8, 24, 6, 11.
$\mathbf{K_{3,2}(4,4;1) \leq 132}$ (4,3)
7; 11, 4, 13, 14, 5, 4, 28, 8, 1, 18, 9, 8, 10, 0, 1, 2, 26, 3, 26, 0, 1, 2^3, 27, 0, 1, 2, 28, 1, 0, 8, 13, 7, 14, 2, 0, 9, 26, 4, 7, 12, 7, 12, 24, 1, 0, 6, 18, 4, 12, 13, 0, 2, 25, 0, 1, 2, 24, 1, 13, 12, 7, 16, 7, 4.
$\mathbf{K_{3,2}(4,5;1) \leq 240}$ (4,4)
12; 1, 22, 5, 8, 18, 4, 25, 2, 3, 26, 1, 5, 2, 28, 12, 9, 2, 5, 0, 21, 2, 9, 15, 1, 4, 5, 33, 4, 0, 8, 31, 10, 6, 10, 31, 14, 6^2, 0, 3, 2, 26, 6, 3, 19, 0^5, 30, 4, 3, 17, 6,

10, 22, 2, 6, 3, 19, 1, 26, 5, 1, 25, 4, 5, 11, 24, 4^2, 19, 5, 28, 1, 5, 21, 2, 9, 11, 24, 1, 6, 20, 2, 29, 4, 3, 20, 5, 6, 15, 2, 6^2, 10, 20, 9, 6, 9, 20, 10, 16, 8, 0, 19, 18, 5, 4^2, 3, 17, 4, 1, 3, 15, 5, 2, 9.

$\mathbf{K_{3,2}(4,5;3) \leq 16}$ (4,4)

14; 114, 6, 235, 8, 197, 238, 212, 6.

$\mathbf{K_{3,2}(4,6;1) \leq 432}$ (4,4)

3, 8; 3, 25, 6, 5, 14, 2, 9, 12, 18, 9, 10, 17, 7, 2, 9, 28, 2, 21, 6, 5, 25, 5, 12, 5, 21, 5, 6, 5, 21, 9, 24, 1, 3, 8, 20, 8, 13, 12, 5, 22, 2, 28, 5, 2, 15, 6, 5, 12, 27, 8^3, 10, 6, 5, 28, 2, 24, 1, 10, 8, 10, 17, 8, 22, 6, 5, 8, 18, 12, 21, 2, 13, 1, 10, 9, 21, 8^2, 20, 2, 28, 5, 6, 7, 21, 2, 9, 22, 8, 9, 10, 8, 5, 8, 25, 7, 12, 21, 6, 2, 5, 22, 5, 25, 5, 2, 9.

$\mathbf{K_{3,2}(5,4;1) \leq 324}$ (4,3)

6, 305; 9, 4, 13, 18, 3, 16, 7, 14, 11, 8, 20, 0, 16, 12, 13, 2, 10, 22, 5, 12, 13, 2, 18, 10, 14, 6, 10, 18, 3, 16, 7, 14, 7, 20, 12, 0, 11, 20, 7, 10, 17, 4, 11, 18, 10, 2, 17, 10, 15, 0, 19, 12, 9, 10.

$\mathbf{K_{3,2}(5,5;1) \leq 639}$ (3,5)

384, 319; 29, 6, 3, 4, 0, 4, 14, 0, 14, 18, 4, 37, 1, 12, 6, 2, 42, 6, 11, 0, 19, 20, 12, 7, 2, 43, 4, 5, 14, 33, 10, 1, 5, 6, 1, 41, 0, 9, 0, 14, 15, 18, 3, 5, 12, 6, 35, 0, 8, 10, 0^2, 37, 6, 0, 15, 6, 13, 1, 3^2, 4, 5, 3, 34, 2, 14, 38, 2, 11, 4.

$\mathbf{K_{3,2}(5,5;3) \leq 32}$ (5,4)

14; 0, 235, 29, 91, 1398, 29, 91, 265, 1444, 0, 72, 0, 120, 0, 64, 0.

$\mathbf{K_{3,2}(5,6;1) \leq 1206}$ (3,5)

31, 352, 319; 8, 28, 12, 8, 18, 13, 19, 2, 21, 0, 18, 6, 2, 37, 1, 9, 6, 2, 26, 22, 0, 7, 5, 2, 36, 4, 0, 24, 19, 21, 0, 21, 11, 2, 5, 0, 12, 30, 13, 3, 8, 13, 34, 6, 0, 27, 11, 4^2, 2, 21, 11, 32, 8, 2, 5, 4, 2, 8, 10, 34, 19, 5, 0, 17, 23, 2.

$\mathbf{K_{3,2}(6,1;2) \leq 27}$ (5,1)

242; 25, 44, 76, 62, 27, 40, 44, 99, 40.

$\mathbf{K_{3,2}(6,3;1) \leq 468}$ (4,2)

3, 144, 7; 15, 17, 8, 17, 13, 17, 10, 12, 8^2, 12, 18, 5, 12, 7, 4, 16, 4, 0, 1, 14, 26, 21, 9, 8, 5.

$\mathbf{K_{3,2}(6,4;1) \leq 864}$ (3,3)

7, 88, 31, 47; 14, 7, 5, 23, 2, 17, 5, 17, 33, 1, 2, 9, 29, 10, 1, 4.

$\mathbf{K_{3,2}(6,5;1) \leq 1656}$ (4,4)

14, 49, 511; 8, 12, 8, 20, 0, 36, 31, 8, 12, 11, 27, 4, 20, 4, 19, 5, 12^2, 29, 10, 12, 30, 12^2, 2, 9, 19, 4, 5, 17, 34, 2, 18, 10^4, 9, 36, 2, 3, 5, 8, 28, 14, 15, 9, 3, 6, 17, 13, 11, 5, 10, 21, 2, 26, 43, 9, 4, 8, 17, 6, 30, 12, 9, 8, 10, 5, 2, 17, 14, 13, 12, 18, 6, 27, 31, 13, 4, 5, 2, 11, 24, 4, 27, 2, 5, 21, 4, 5, 10.

$\mathbf{K_{3,2}(7,0;1) \leq 186}$ (6,0)

363; 14, 15, 6, 22, 1, 9, 8, 0^2, 40, 6, 15, 12, 16, 6, 12, 9, 12, 6, 22, 11, 0, 21, 1, 4, 17, 26, 10, 1, 0^3, 24, 0^2, 27, 1, 3, 42, 17, 4, 1, 19, 4, 12^2, 11, 3, 17, 0^2, 36, 1, 9, 19, 4, 30, 4, 5, 0, 21, 12.

$\mathbf{K_{3,2}(7,1;1) \leq 333}$ (5,1)

234, 149; 17, 12, 7, 0, 22, 0, 13, 6, 27, 28, 8, 0, 24, 4, 26, 0, 19, 6, 25, 0, 10, 8,

5, 14, 10, 8, 16, 8, 0, 26, 22, 26, 0, 5, 30, 7, 0.
$\mathbf{K_{3,2}(7,2;1) \leq 648}$ (5,2)
468, 299; 0, 32, 33, 2^2, 1, 12, 1, 30, 1, 10, 18^2, 2, 12, 1, 36, 8, 1, 15, 22, 2, 24, 1, 16, 26, 2, 10, 0, 17, 32, 1, 41, 2, 6, 2, 41, 1, 6, 1, 23, 17, 0, 10, 2, 26, 12, 1, 28, 2, 22, 15, 1, 8, 27, 1, 21, 2, 18^2, 10, 1, 38, 1, 9, 2, 33, 32, 17, 1, 12, 1.
$\mathbf{K_{3,2}(7,2;3) \leq 33}$ (6,2)
1452; 649, 4, 154, 155, 400, 118, 47, 494, 10, 38, 119.
$\mathbf{K_{3,2}(7,6;1) \leq 8640}$ (4,4)
13, 0, 561, 447, 31; 17, 0, 5, 2, 19, 4, 14^2, 27, 21, 14, 5, 2, 29, 27, 10, 30, 4, 9, 16, 26, 27, 32, 4, 13, 16, 10, 18^2, 0, 7, 10, 24, 43, 10, 31, 0, 1, 9, 25, 4, 29, 11, 32, 4, 17, 4, 16, 2, 9, 22, 20, 5, 14^2, 36, 8, 26, 16, 10, 16, 4, 29, 9, 17, 5, 32, 21, 13, 22, 20, 7, 28, 5, 2, 19, 4, 9, 17, 24.
$\mathbf{K_{3,2}(8,1;3) \leq 45}$ (7,1)
2178; 335, 3, 290, 583, 8, 297, 6, 613, 738, 8, 297, 6, 620, 97, 8.
$\mathbf{K_{3,2}(8,4;1) \leq 6480}$ (4,3)
7, 24, 23, 231, 215; 23, 9, 15, 8, 9, 23, 18, 10, 12, 1, 6, 10, 17, 16, 31, 6, 10, 18, 34, 1, 18, 17, 44, 15, 17, 4, 13, 17, 18^2, 4, 5, 10, 5, 18, 9, 34, 10, 9, 20.
$\mathbf{K_{3,2}(8,5;2) \leq 1620}$ (5,4)
15, 1904, 303, 959; 0, 13, 183, 193, 169, 49, 202, 124, 180, 50, 205, 204, 36, 130, 195^2, 117, 55, 186, 201, 55, 182, 137, 1, 186, 42, 181, 183, 194, 0.
$\mathbf{K_{3,2}(8,5;3) \leq 324}$ (6,4)
12, 563, 4607; 2988, 6, 229, 12, 229, 6, 2033, 4, 325, 14, 325, 4, 1976, 24, 217, 18, 217, 24.
$\mathbf{K_{3,2}(9,2;1) \leq 4860}$ (5,2)
480, 75, 179, 59; 2, 16, 0, 14, 23, 0, 99, 40, 6, 0^3, 8, 18, 12, 0^3, 64, 26, 4, 0^3, 13, 26, 17^2, 33, 7, 18, 13, 4, 3, 36, 42, 0^3, 17, 20, 10, 8, 3, 23, 54, 5, 15, 13, 29, 20, 16, 4, 10, 4, 61, 0^3, 15.
$\mathbf{K_{3,2}(9,3;2) \leq 1215}$ (5,3)
960, 151, 359, 119; 24, 404, 86, 1, 69, 21, 106, 272, 28, 157, 129, 372, 6, 178, 3.
$\mathbf{K_{3,2}(9,4;1) \leq 17496}$ (4,3)
7, 232, 47, 175, 47, 119; 34, 20, 1, 32, 20, 1, 31, 21, 26, 25, 31, 27, 2, 1, 0, 21, 3, 2, 1, 0, 29, 19, 2, 1, 0, 29, 7, 21, 46, 13, 41, 13, 10, 20, 9, 24.
$\mathbf{K_{3,2}(10,1;1) \leq 7047}$ (6,1)
8, 63, 575, 625; 13, 10, 36, 6, 26, 14, 24, 11, 12, 9, 24, 3, 8, 25, 16, 12, 5, 10^2, 38, 2, 33, 18, 12, 13, 2, 44, 0, 9, 31, 14, 16, 4, 26, 24, 13, 2, 24, 15, 0, 42, 18, 12, 16, 2, 25, 2, 23, 17, 10, 14, 21, 12, 32, 10, 16, 19, 6, 3, 2, 36, 18, 24, 13, 0, 33, 2, 20, 29, 6, 14, 12, 26, 21, 14, 21, 14, 12, 16, 6, 16, 19, 8, 11, 23, 4, 10.
$\mathbf{K_{3,2}(10,2;1) \leq 13122}$ (5,2)
160, 179, 127, 219, 79; 5, 26, 28, 0^2, 60, 1, 0, 65, 20, 0, 42, 0, 4, 31, 0, 82, 4, 0^2, 25, 19, 14, 1, 0, 26, 0, 84, 55, 0, 20, 0, 18, 0, 20, 26, 0, 17, 0, 12, 29, 2, 7, 23, 6, 15, 41, 0, 5, 51, 4, 0, 18, 0.
$\mathbf{K_{3,2}(12,1;1) \leq 52488}$ (6,1)

8, 207, 287, 35, 107, 539; 20, 14, 18, 10, 3, 54, 0, 21, 2, 13, 4, 84, 14, 19, 8, 3, 64, 8, 23, 4, 2, 28, 2, 4, 41, 8, 3, 80, 2, 4, 25, 2, 65, 10, 3, 22, 14, 22, 4, 2, 66, 8, 3, 18, 2, 89, 10, 40, 4, 2, 26, 12, 21, 4, 69, 2, 4, 25, 12, 7, 0, 71, 8, 7, 10, 21, 51, 8, 3, 21, 10, 3.

$\mathbf{K_{4,3,2}(1,1,8;2) \leq 48}$ (1,1,6)

52, 3; 50, 42, 36, 105, 70, 40, 66, 32, 110, 17, 108, 66.

$\mathbf{K_{4,3,2}(1,3,4;2) \leq 30}$ (1,3,3)

7; 11, 31, 55, 31, 15, 63, 58, 33, 92, 87, 40, 81, 87, 44, 88.

$\mathbf{K_{4,3,2}(1,4,3;2) \leq 36}$ (1,3,3)

96; 25, 63, 116, 51, 31, 130, 57, 31, 120, 43, 63, 110.

$\mathbf{K_{6,3}(1,6;1) \leq 288}$ (1,5)

121; 1^2, 38, 3, 30, 16, 12, 2, 33, 14, 19, 0, 11, 18, 11, 3, 35, 11, 7, 19, 3, 16, 6, 36, 0, 30, 16, 12, 2, 22, 7, 5, 23, 7, 15, 7, 31, 19, 9, 11, 18, 2, 12, 16, 31, 10, 21, 10, 1, 22, 13, 27, 12, 6, 9, 15, 13, 24, 3, 16, 12, 11, 13, 4, 40, 9, 2, 25, 4, 5, 31, 3, 19, 10, 11, 20, 1, 38, 21, 9, 14, 12, 11, 13, 7, 14^2, 20, 10, 19, 20, 8, 34, 5, 19, 5.

$\mathbf{K_{6,3,2}(1,1,6;1) \leq 96}$ (1,1,5)

28; 24, 2, 5, 12, 6, 12, 10, 8, 29, 2, 10^2, 12, 6, 12, 10, 35, 0, 5, 3, 10, 19, 3, 4, 24, 0, 28, 8, 14, 4, 14, 8, 3, 2, 8, 2, 38, 2, 27, 4, 10, 0, 10, 0, 24, 16, 11, 22.

Table V. Partitions of codes.

$\mathbf{(3,2;0,6;12)1}$

010122221010

$\mathbf{(3,2;0,10;118)2}$

2001221210202111110120122000022021010020221001210200121212122011
0021002010112121001222121210120221210211201022011200002

$\mathbf{(3,2;1,4;48)0}$

044251133115244015503224422305512331400550041332

$\mathbf{(3,2;1,6;24)1}$

020020001121111220120221

$\mathbf{(3,2;1,8;84)1}$

2212000112020100202111200112210110000222011201220121101202102 20
211022102102110220110

$\mathbf{(3,2;1,9;160)1}$

1200222111200011012012022011102200210121022022120010202202020 21
0200111220210101022211102212111000211222202200001211021021101 12
1212100020012001021221001101112012

$\mathbf{(3,2;2,2;36)0}$

043152251340154230032451235041140532

$\mathbf{(3,2;2,7;30)2}$

011222100110022220011100222011

$\mathbf{(3,2;3,2;130)0}$

{01}77{01}655{24}{24}3363{46}{26}5710350176{01}{01}6{57}245324{3
7}54{46}3301{27}{27}1050271475667344{35}741{36}201{56}205{26}233
01{47}{47}10517402576637227{35}204{36}104{56}1

(3, 2; 3, 5; 92)1

01111100210201120221200112221022001020121020210212100221001122 2
222111022210012010011122000 10

(3, 2; 3, 7; 320)1

1010222110121202120102210112220201121112222100100010212012012 20
2122101100112112112210002101010020222221021002111002122212001 00
1112000202101120120020202011100122100222000202212010022012120 12
0022001101111112121211210012010022021200122010010102201010200 11
0210112112011221112122221220101102021110200020020202212001122 02
20100

(3, 2; 3, 7; 60)2

001122221100122101100110220011110022222220020110112200002211

(3, 2; 4, 1; 18)1

101202200121120201

(3, 2; 4, 4; 132)1

0000002222112211111211001211200112000222201001122201011020121 02
1102200112211012200110022222200221121001022010021120120021211 22
001100

(3, 2; 4, 5; 240)1

1220212101102122010221100222111021220110000002002201021012210 21
0012222012001022112001102121122012210200022120210200210210221 10
2111011222212101020210122010210011211000102010010221120022111 02
022000101122111200221101002221001022012111011020201

(3, 2; 4, 6; 432)1

0002011201022022022100202102222102011120120110010102211112201 00
1011101100210220222220220201111010102200220112110111001212202 10
0022100010121202210220102000102112100210211001101112101210221 22
2222010120100122222200220212200100101201101101201221220001112 21
2102202202011211001220220220002110022101210101102001120100221 12
2012212122000201210011100011100122011201112221220200001201010 10
110222002221112210202212111120000012201111210220121012

(3, 2; 5, 2; 96)1

0100210202211021012110110201021221022011102202021020210112022 01
111201020120122101200222001201120

(3, 2; 5, 3; 168)1

1122022021001022110001102021211002012221211010222121000120110 00
1212221122100122101202012102010211002010210121020212121001211 20
002001022001020121121001020012220201122211

(3, 2; 5, 3; 168)1

1200213333321032013333310233201310201232013310201232103333321 03

33201012102333201312002132103330121021023332013333320131020123310231020123210332013333321032013333312002 1

$(\mathbf{3}, \mathbf{2}; \mathbf{5}, 4; \mathbf{324})\mathbf{1}$

22010012111110002120200212021012110002021111200010021202102111021202212221202012202201202211110200000111122201012001211010022111022220112202202122221010200022012021122100011020011112111002201102112001001220202121200110222010120002211201021020111021202000011112221022102011201011022120021112212001012011022002012221220100110 0

$(\mathbf{3}, \mathbf{2}; \mathbf{5}, 4; \mathbf{336})\mathbf{1}$

1103322230010123231032103201101322302301233210012310101002233123013210123320100322311032002311132230230132103210103223103201231010123320032231103201013210022331321013332220101010230132103213322000231111320023230112332010103223101322301000321102332110320023113211320013322010002311323213322023100123010023113202332110321001320112233321 00

$(\mathbf{3}, \mathbf{2}; \mathbf{6}, \mathbf{0}; \mathbf{73})\mathbf{1}$

11200221100102210221012000102112021110101022221021220001022122112021001 02

$(\mathbf{3}, \mathbf{2}; \mathbf{6}, \mathbf{1}; \mathbf{132})\mathbf{1}$

21011002111102202022100102120000212112002102122102112011120100211220200110212012120211200021102020212021001122200110102111002200122 2

$(\mathbf{3}, \mathbf{2}; \mathbf{6}, \mathbf{1}; \mathbf{31})\mathbf{2}$

0012012120021212100021120120210

$(\mathbf{3}, \mathbf{2}; \mathbf{6}, \mathbf{3}; \mathbf{78})\mathbf{2}$

0121212012121200210120212001212001010020{02}21{01}100221{01}1222101202122{01}1002101{01}{02}20210

$(\mathbf{3}, \mathbf{2}; \mathbf{7}, \mathbf{0}; \mathbf{186})\mathbf{1}$

210000012221202121210100210021212212000122212121102020010100121010212121021202020100022012100011100011221020120210101012222111210021220210201212201201220120201002121211012212120010010110

$(\mathbf{3}, \mathbf{2}; \mathbf{7}, \mathbf{1}; \mathbf{54})\mathbf{2}$

101202101202101202212010212010212010020121201021021012

$(\mathbf{3}, \mathbf{2}; \mathbf{8}, \mathbf{0}; \mathbf{81})\mathbf{2}$

102210021210021102021102210021102210102210021210021102210021102021102210102210021

4 Conclusions

An updated version of the table of upper bounds for binary/ternary mixed covering codes has been presented. All but 2 of the new bounds given in Table II have been found by the authors; $K_2(12,1) \leq 380$ was reported to us by G. Exoo [8], and $K_{3,2}(4,7;3) \leq 45$ by E. Madsen [23]. The latter code was published as a football pool system in the Danish newspaper Herning Bladet on November 20th, 1991.

Many of the bounds can probably still be improved. To keep future versions of the main table up-to-date, we would like to get to know of any improvements on the upper bounds (including complete descriptions of constructions). Please send them to the first author by ordinary mail or by electronic mail (Patric.Ostergard@hut.fi).

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