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CONSTRUCTIONS OF MIXED COVERING CODES

Patric R. J. Östergård

Digital Systems Laboratory
Department of Computer Science
Helsinki University of Technology
Otaniemi, Finland

Helsinki University of Technology
Department of Computer Science
Digital Systems Laboratory
Otaniemi, Otakaari 1
SF–02150 ESPOO, FINLAND

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Abstract: In this work construction methods for so called mixed covering codes are developed. There has been considerable recent growth in the interest in covering codes. They have the property that all words in the space are within a given Hamming distance, called the covering radius, from some codeword.

Traditionally, mainly spaces where all coordinates have the same arity (usually 2 or 3, that is, they are binary or ternary) have been discussed. In this work we consider mixed spaces $F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$, where the coordinates can be of varying arities. The approach is very general, no restrictions are set upon m and the arities q_i.

The construction methods consist of generalizations of known constructions for covering codes and some completely new constructions. They are divided into three classes; direct constructions, constructions of new codes from old, and the matrix method. Through these constructions upper bounds for the minimal number of codewords in a code covering a certain space with a certain covering radius ($K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R)$) are improved. The results include a new record-breaking binary code, proving $K_2(12, 2) \leq 78$.

Keywords: Covering code, covering radius, football pool problem, mixed code, simulated annealing.

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Helsinki University of Technology
Department of Computer Science
Digital Systems Laboratory
Otaniemi, Otakaari 1
SF–02150 ESPOO, FINLAND

Phone: $\frac{90}{+358\text{–}0}$ 4511
Telex: 125 161 htkk sf
Telefax: +358–0–465 077
E-mail: lab@hutds.hut.fi

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1 Introduction

Assume that somebody asks you to answer the four questions in Figure 1 (something like this is not very far-fetched nowadays, when "trivial pursuit"-like games are very fashionable). As can be seen, three alternative answers are given for each question. Further assume that you are not limited to one quadruple of answers. This restriction outrules for example school exams, however, in many betting games and competitions this is allowed or can be circumvented by using the names of other family members, etc.

When were the following persons born?

1. Abraham Lincoln.	a) 1775,	b) 1784,	c) 1809.
2. Leonardo da Vinci.	a) 1333,	b) 1452,	c) 1812.
3. Urho Kekkonen.	a) 1890,	b) 1895,	c) 1900.
4. Sir Isaac Newton.	a) 1642,	b) 1717,	c) 1822.

Figure 1: Four questions.

If you do not possess any knowledge at all in the area of these questions, and you want to give a set of answers that always contains one correct quadruple whatever the correct answers are, all possible quadruples have to be given. The number of these is $3^4 = 81$. However, assume that 1 incorrect answer is allowed, that is, there must in any case be at least one quadruple with 3 correct answers. This problem is a covering problem, and a set of quadruples that fulfills the requirements is called a covering code with covering radius 1.

In the general case, the number of alternative answers for each question is denoted q, and the number of questions is denoted n. The number of all possible n-tuples is then q^n, and if there must be at least $n - R$ correct answers in an n-tuple, this is fulfilled by a covering code with covering radius R.

Within the mathematical literature covering codes originated in 1948, when Taussky and Todd [61] discussed the problem group-theoretically. Even earlier, however, such codes had been constructed for football pools. Many results, obtained by "professional amateurs", have remained unknown to the scientific society for half a century [37, 66]. During the years many contributions to the research have appeared in different journals within the areas of coding theory and combinatorics. The main interest has concerned binary codes ($q = 2$) and ternary codes ($q = 3$) of covering radius 1 (the so called *football pool problem* [40]). There has been considerable recent growth in the interest in the area [12, 14, 24].

There are many directions of research within the area of covering codes. There are even many types of covering codes. In many studies (e.g., [12, 24]), only *linear* covering codes are considered. Linear codes have the property that they form a linear subspace of the original space. If we in our previous example require that there are always at least μ (≥ 2) quadruples with at least $n - R$ correct answers, we have to use a *multiple covering* [11, 65]. It is not difficult to think of other similar variants and special cases.

Our work explicitly deals with the construction of codes that have the covering property. The construction of such a code is indeed no problem. For example, by taking all possible quadruples in our example, the set of answers will certainly fulfill (almost) any covering requirements. A real problem, which we consider, is to find the minimal number of codewords that has a certain covering property. It turns out that if 3 correct answers are required in our example ($R = 1$), then exactly 9 quadruples are needed [38, Theorem 4]. Exact values of the minimal number of codewords needed are known only in some rare cases. If the number of questions, all with 3 alternatives, are 5, there are 27 quintuples guaranteeing 4 correct answers (again $R = 1$) [40, Theorem 2.1]. However, for 6 questions and $R = 1$ it is known that the minimal set of answers has a cardinality of 63–73 (see [69] for references). This means that there is a set of 73 sextuples that guarantees 5 correct answers, it is not known whether there are 72 such sextuples, however, it is known that 62 sextuples will not do. In most cases only upper and lower bounds for these cardinalities are known, and it is in many cases very improbable that the exact values ever will be proved. Upper bounds are improved by constructing new record-breaking codes.

Traditionally only covering codes where all the coordinates are of the same arity have been considered. However (cf. Figure 1), if you know that Leonardo da Vinci did not live in the 19th century, the third alternative for the second question can be excluded. The same situation also occurs if the original number of alternative answers for the questions vary. We then have to consider a *mixed space* and *mixed covering codes*. It is clear that if alternatives are excluded, then the cardinality of a minimal set of answers needed decreases or remains the same. Continuing the discussion of our example, when three of the questions have three alternatives and one question has two alternatives as described above, it is not known whether there is a covering set of cardinality less than 9 [37]. Further, if two of the questions have three alternatives and two questions have two alternatives, the minimal covering set consists of 6 quadruples [37]. Recently some papers discussing these mixed covering codes more or less in a general context have been published [37, 52, 63, 68]. Earlier research has mainly concentrated on *perfect* mixed covering codes, see [64] for a reference list of these contributions.

The constructions of mixed covering codes can roughly be divided into two groups. Covering codes can either be constructed from scratch or known covering codes can be used and combined to get new ones. The latter methods are very nice, but cannot be used to give record-breaking codes so often. The former methods can further be divided into combinatorial methods and probabilistic methods. As for the probabilistic methods, Simulated Annealing (SA) has turned out to be an indispensable aid. In the probabilistic search it is often necessary to define the structure of the code that is searched for. This can be accomplished by the matrix method, which is the last but maybe the most interesting (and useful) method presented in the work.

The structure of the report is as follows. In Chapter 2 the formal definitions and the basic theory of covering codes are given. Basic mathematical tools, including SA, that are needed later on are also discussed. The construction methods are presented in Chapter 3 and, finally, we conclude the report in Chapter 4.

2 Preliminaries

In this section we give an introduction to covering codes, and discuss the mathematics needed for these and for understanding the theory of the rest of the report. As a matter of fact, most of the proofs are based on combinatorial arguments, which anybody that is able to interpret the mathematical notations can comprehend.

The following details should be borne in mind while reading the text. In the definitions, we use the word *if* meaning *if and only if*. For p a prime, the prime powers of p are $p^1, p^2, \ldots$, i.e., a prime is itself a prime power.

2.1 Algebraic Systems

In Figure 1, the set of possible answers is for each question $\{a, b, c\}$. Generally, in coding theory, the values for a q-ary coordinate are integers taken from the set $\{0, 1, \ldots q-1\}$. In order to be able to operate on these values, they are assumed to be representatives of certain *algebraic systems*. The three main types of algebraic systems are *groups*, *rings* and *fields*. We here give the definitions for such algebraic systems, for a more extensive treatment of these and other concepts of abstract algebra, the reader is referred to [21].

Definition 1 *Let G be a nonempty set and $*$ an operation defined on G. Then the pair $(G, *)$ is a* group, *if*

1. *for all $g, h, k \in G$, $g * (h * k) = (g * h) * k$* (associativity),
2. *there is an $e \in G$* (unit-element), *such that for all $g \in G$, $g*e = e*g = g$, and*
3. *for all $g \in G$, there is an $h \in G$* (inverse element), *such that $g * h = h * g = e$.*

The integers with addition as operation $(\mathcal{Z}, +)$ is a group with unit-element 0. Addition is commutative, i.e., for all $g, h \in G$, $g * h = h * g$. Commutative groups are called *Abelian groups*.

Definition 2 *Let R be a nonempty set and $+$ and $\cdot$ operations defined on R. Then the triple $(R, +, \cdot)$ is a* ring, *if*

1. *$(R, +)$ is an Abelian group (with unit-element 0),*
2. *the operation $\cdot$ is associative, and*

3. *for all $r, s, t \in R$, $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(r + s) \cdot t = r \cdot t + s \cdot t$* (distributivity).

It is easily seen that $(\mathcal{Z}, +, \cdot)$ is a ring. This system is, however, not a field, which we define in the following.

Definition 3 *Let F be a nonempty set and $+$ and $\cdot$ operations defined on F. Then the triple $(F, +, \cdot)$ is a* field, *if*

1. *$(F, +)$ is an Abelian group (with unit-element 0),*
2. *$(F \setminus \{0\}, \cdot)$ is a group (with unit-element e), and*
3. *distributivity holds.*

The rational numbers with addition and multiplication as operations $(\mathcal{Q}, +, \cdot)$ is an example of a field. Both the integers and the rational numbers are infinite sets. The corresponding systems are then called *infinite*. Within the theory of covering codes we are only interested in systems where the basic set (G, R, or F) is finite, these are called *finite groups, finite rings*, and *finite fields*. Such a system is said to be of order n if its basic set consists of n elements.

Most interesting are finite fields. It is known that finite fields of order q exist, if and only if q is a prime power. It is even known that for a fixed prime power q, the corresponding finite field is unique. Such a field, also called a *Galois field*, is denoted by $\mathcal{F}_q$ or GF(q). If q is a prime, we can take $\mathcal{F}_q$ to be the set $\{0, 1, \ldots, q-1\}$ and let $+$ and $\cdot$ be the corresponding arithmetic operations modulo q. If $q = p^n$, where p is a prime and $n \geq 2$, the operations are not that easily defined, the reader is referred to [21] for more information on that case.

In the rest of the report F_q denotes the finite field $\mathcal{F}_q$ if q is a prime power, otherwise the ring of integers modulo q.

2.2 Introduction to Covering Codes

Most monographs on coding theory start with presenting a communication system, discussing the need for coding data that is sent through some type of channel. However, in case of covering codes, the main applications are not of this type (see Subsection 2.2.1 for applications of covering codes), so we here just give the main definitions.

We here consider arbitrary *mixed spaces* $F = F_{q_1} F_{q_2} \cdots F_{q_n}$, where $n \geq 1$ and $q_i \geq 1$. A *word* in this space is an n-tuple $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in F_{q_i}$.

(Such a word is usually written in the shorter form $x_1 x_2 \cdots x_n$.) We say the q_i:s are the *arities* of the respective spaces. A *code* C is now a nonempty subset of F; $C \subseteq F$. The elements in C are *codewords*, and the *length* of the code is n. Actually, in some cases it is necessary to define codes as *multisets*, i.e., to allow multiple occurrences of codewords in the code. However, in our study this is not required. To get a shorter form for F, we group the coordinates together, like $F_2 F_2 F_2 = F_2^3$. We can now write the space in the following form, where the value of q_i is not necessarily the same as the value of q_i in F: $H = F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}, m \geq 1, n_i \geq 0, q_i \geq 1$. (In the rest of the report, H always refers to this space.) It will later be clear why we allow $n_i = 0$. If $m \geq 2$ and not all the q_i are the same, we have a *proper* mixed space, but unless specifically stated we do not require this further on.

The *Hamming distance* $d(x,y)$ between two words $x, y \in H$ is the number of coordinates in which they differ. The *weight* of a word $x \in H$ is $wt(x) = d(x, 0)$, i.e., the number of nonzero coordinates $(0 = (0, \ldots, 0))$. For $x, y \in H$, we then have that $d(x,y) = wt(x-y)$, where $x - y$ stands for coordinatewise subtraction within the respective algebraic systems. This distance function is a *metric*, since for all $x, y, z \in H$

1. $d(x,y) \geq 0$,
2. $d(x,y) = 0 \Leftrightarrow x = y$,
3. $d(x,y) = d(y,x)$, and
4. $d(x,z) \leq d(x,y) + d(y,z)$.

In the binary case an n-dimensional space can be displayed as a n-dimensional hypercube. There is now an one-to-one mapping between the nodes and the words in the space, so that two nodes in the hypercube are adjacent exactly when the corresponding codewords differ in one coordinate (is at Hamming distance 1 from each other). In Figure 2 this is illustrated for $n = 4$.

A code $C \subseteq H$ is said to cover H with *covering radius* R if every word in H is within Hamming distance R from some codeword in C, and if this is not fulfilled for $R' = R - 1$. If such a mixed code has M codewords, we call it a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code. The number of codewords M is called the *cardinality* of the code. The set of codewords must be nonempty, so the cardinality has to be positive. If a code is a linear subset of the space, it is called a *linear code*. A linear code that has q^k (q is a prime power) codewords and covers F_q^n with covering radius R is called a $[n,k]R$ code. We further denote

$$K_{q_1, q_2, \ldots, q_m}(n_1, n_2, \ldots, n_m; R) =$$
$$\min\{M \mid \text{there is a } (q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R \text{ code}\}.$$

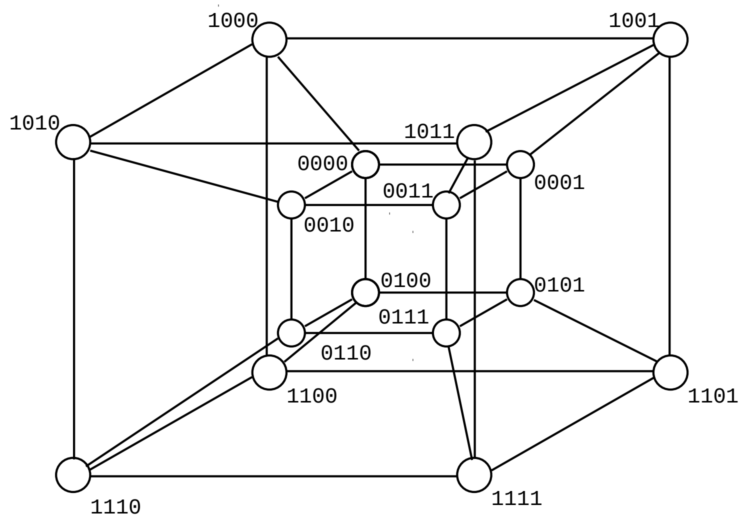

Figure 2: Hypercube of dimension 4.

In the consideration of the nonmixed case, the notation $K_q(n, R)$ has been used, where the index q can be omitted in the binary case ($q = 2$). Other notations that have been used in the literature include $\sigma(n, k) = K_k(n, 1)$, $\sigma_n = K_3(n, 1)$ (for example in [45, 67]) and $\gamma(n, k, s) = K_k(n, n - s)$ [7, 8]. In the mixed case, $K(t, b, R) = K_{3,2}(n_1, n_2; R)$ has been used in [37, 52].

Very few exact values for these functions are known, so efforts have mainly been concentrated on improving lower and upper bounds. By considering the total number of words in a space and the number of words covered by a codeword in the space, the following *sphere covering bound* is obtained.

Theorem 1 $K_{q_1,q_2,\ldots,q_n}(1, 1, \ldots, 1; R) \geq (\prod_{i=1}^{n} q_i)/V(R)$, *where* $V(R) = 1 + \sum_{j=1}^{R} \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} (q_{i_1} - 1) \cdots (q_{i_j} - 1)$.

Proof: We first show that $V(R)$ is the number of words within distance R from a codeword. There is one word at distance 0 from the codeword, i.e., the codeword itself. The number of words at distance j from the codeword ($1 \leq j \leq R$) is obtained by first considering any possible set of j coordinates $1 \leq i_1 \leq \cdots \leq i_j \leq n$. The number of words that differ from the codeword exactly in these coordinates is $(q_{i_1} - 1) \cdots (q_{i_j} - 1)$. By summing over all distances from 1 to R, we get that $V(R) = 1 + \sum_{j=1}^{R} \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} (q_{i_1} - 1) \cdots (q_{i_j} - 1)$. Since all words must be covered by some codeword, and the number of words in the space is $\prod_{i=1}^{n} q_i$, it is clear that $V(R) \cdot K_{q_1,q_2,\ldots,q_n}(1,1,\ldots,1;R) \geq \prod_{i=1}^{n} q_i$, from which the theorem follows. □

Closely related to this theorem is the concept of *density*.

Definition 4 *Let $V(R)$ be defined as in Theorem 1. The density of a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code is $M \cdot V(R)/\prod_{i=1}^{n} q_i$.*

From Theorem 1 it follows that the density is always ≥ 1. The density can be seen as a meter of the quality of the covering code. The closer the density is to 1, the better is the code. If the density is 1, i.e., the bound in Theorem 1 is attained, then the corresponding code is called *perfect*. Then all the words in the space is within distance R from exactly one codeword. Such a code is then simultaneously a *packing code*. Now, for example $C = \{000, 111\}$ is a perfect $(2,3,2)1$ binary code. It is easily seen that all words in $F_2^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$ are within Hamming distance 1 from C. The words within distance 1 from 000 are $C_1 = \{000, 001, 010, 100\}$ and the words within distance 1 from 111 are $C_2 = \{011, 101, 110, 111\}$. Further, $C_1 \cup C_2 = F_2^3$ and $C_1 \cap C_2 = \phi$. It follows that $K_2(3,1) = 2$.

In Figure 3 it is showed that $K_2(4,1) \leq 4$. (All nodes are either black or have an adjacent black node.) Using Theorem 1 we obtain $K_2(4,1) \geq 16/5 = 3.2$. Thus $K_2(4,1) = 4$.

This report deals with upper bounds. By explicitly constructing covering codes upper bounds on $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R)$ are improved. In the sequel the main theorems consist of such upper bounds. This does in no way interfere with our intention of presenting construction methods, since all proofs are indeed constructive. The first paper on construction methods for mixed covering codes is a paper by Hämäläinen and Rankinen [37]. They, however, discuss only binary/ternary mixed codes, i.e., they give upper bounds on $K_{3,2}(n_1, n_2; R)$. Some of the constructions to be presented are generalizations of their results. Binary/ternary mixed codes are interesting as they can be used in football pool systems. In football pools there are three alternatives for each match: home team wins, home team loses, draw. Two alternatives remain if the gambler is convinced of the impossibility of one of these. A Finnish football pool coupon (front and back sides) is illustrated in Figure 4.

An interesting feature when betting on the Finnish football pools is the possibility of using some so called reduced systems. These are marked A–F in the coupon in Figure 4, and are called H-systems. As a matter of fact, these are mixed covering codes of the following types: A) $(3,2;3,7;60)2$, B)

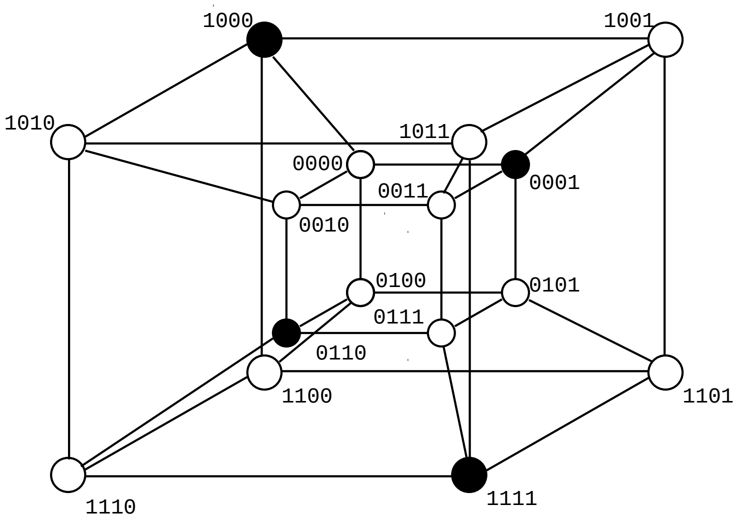

Figure 3: Covering code of length 4.

$(3, 2; 8, 0; 81)2$, C) $(3, 2; 9, 0; 222)2$, D) $(3, 2; 3, 8; 432)2$, E) $(3, 2; 10, 0; 729)2$, and F) $(3, 2; 4, 8; 1296)1$. Out of these, B and F are the best known codes with the respective parameters, and D and E are multiple coverings. The codes of types A and C have been improved; Hämäläinen has found a $(3, 2; 3, 7; 56)2$ code [36], and the present author has found a $(3, 2; 9, 0; 219)2$ code [69]. The $(3, 2, 4, 8; 1296)1$ code is remarkable, since it has the smallest known density (1.0625) of all proper binary/ternary mixed codes of length at most 13. It is known that no perfect proper binary/ternary mixed codes exist (see [64, Corollary 2]), so the density is in any case > 1.

2.2.1 Applications of Covering Codes

In the literature, the following possible applications have been mentioned through the years. Of these, the connection to football pools is by far the most frequently mentioned.

Football Pools and Quizzes This application has been shown in the Introduction and in the previous subsection. As a matter of fact, these codes (systems) are widely used especially among football pool gamblers in the Nordic

Figure 4: Finnish football pool coupon.

countries. Results on the football pool problem with parameters $q = 3$, $R = 1$ have been published especially in the *Journal of Combinatorial Theory Series A* [6, 20, 40, 45, 62, 67].

Distribution of Resources Let us consider a hypercube computer of dimension n. The number of nodes (processors) is then 2^n. The nodes can then be given binary n-tuples such that there is an immediate connection between two nodes exactly when the corresponding n-tuples differ in one coordinate. (Figure 2 can be seen as a hypercube of dimension four.) In case of expensive resources that are attached to the hypercube, they cannot be attached to each node. A reasonable criterion is now that the resources are within a certain distance from each node. This is exactly the covering problem!

This problem was discussed by Livingston and Stout in [53]. They seemed, however, to be unaware of the theory of covering codes, since no connections to covering codes were made. The results in the paper are also quite elementary compared with the state of the art at that time (1988).

Coding of Speech In [24] Graham and Sloane mention that one of their reasons for starting investigating covering codes was a question raised by their colleagues Atal and Schroeder concerning efficient coding of speech. One subproblem in that area is to find the best way to place M points "uniformly" on the surface of a sphere in an n-dimensional Euclidean space. An approximate solution can then in certain cases be obtained by taking the codewords of a good covering codes.

It is not clear whether covering codes has been used in practise for coding of speech, since this problem has as far as we know not been mentioned in connection with covering codes since [24].

Covering Radius as a Parameter for Codes The covering radius of a code is a geometric parameter that in some situations give important information about the performance of the code. The following list is extracted from [12]. The results hold for linear codes, which are the codes considered in that paper.

- If a code is used for data compression, the covering radius is a measure of the maximum distortion [5]. If it is used for error correction, the covering radius is a measure of the maximum weight of a correctable random error.

- The covering radius is an important parameter for lattices in Euclidean space [15, Ch. 23]. These lattices have applications to quantization and to coding for the Gaussian channel (see [15, Ch. 21] and [16]).
- Covering radius plays an important role in a decoding procedure presented by Levitin and Hartmann in [46].

2.3 Simulated Annealing

Combinatorial optimization problems typically involve a cost function to be minimized (maximization is obtained by minimizing the negative function). A well known approach to such problems is named *iterative improvement*, working in the following way. Starting from an initial configuration one of the "neighbouring" configurations is explored. This new configuration is accepted if the value of the cost function is less for it than for the present configuration. This evaluation is repeated until no further improvements can be made.

The drawback of such a *greedy* algorithm is that it gets stuck in local minima, since only downhill moves are accepted. One way to cope with this problem is to repeat the algorithm for a number of times from different initial configurations, and hope that at least one iteration will end up in a global minimum. Another way is to allow some uphill moves also; this is one of the basic features of the *Simulated Annealing* (SA) optimization method.

The SA algorithm was independently published by Kirkpatrick, Gelatt Jr. and Vecchi [42], and Černy [9] in the early 1980's. The method is motivated by an analogy to physical processes. When a material is crystallized from the liquid phase, it tries to attain a state where the energy is at a minimum. This cooling (annealing) must be done very slowly to obtain a highly ordered, low-energy final state. At each temperature the likelihood of its being in a given state is given by the Boltzmann distribution for that temperature. For an optimization problem we introduce a parameter T simulating the temperature of a physical process. The cost function $E(s)$ to be minimized corresponds to the energy of a process.

During the decade that has passed since the introduction of this algorithm, the interest in it has been steadily growing. Several monographs discussing simulated annealing have already been published. For extensive consideration of the history and the mathematical theory of the algorithm the books [1, 44] are warmly recommended. SA has successfully been applied to problems such as VLSI design [17], routing problems [23], and the traveling salesman problem [9]. Typical for most of the problems within these areas is the fact that no algorithm solving them in polynomial time is known, i.e., they are NP-complete problems [22] or even harder. We can then use simulated annealing to obtain a near-optimal solution within reasonable time.

The general form of the SA algorithm is showed in Figure 5 (in pseudo-code).

Start from $T := T_0$ and a random initial state s
while stop criterion not satisfied **do**
  begin
  while inner loop criterion not satisfied **do**
    begin
    Generate new state s'
    $\Delta E := E(s') - E(s)$
    if $\Delta E \leq 0$ **then** $s := s'$
    if $\Delta E > 0$ **then** $s := s'$ with probability $e^{-\Delta E/T}$
    end
  Update T
  end

Figure 5: Simulated annealing algorithm.

The generation of new states and the energy function are both problem-specific. These parameters are discussed in Section 3 when the different design methods are described. The general parameters are the initial temperature, the stop criterion, the inner loop criterion, and the update of the temperature. The choice of these parameters is referred to as the *cooling schedule*.

Much research has been done in finding optimal parameters, and several cooling schedules have been presented through the years. However, the most advanced cooling schedules usually involve lots of statistical calculations, which sometimes are so time-consuming that the overall performance is deteriorated. We have used the following very simple cooling schedule, which turned has out to be surprisingly good.

- **Initial temperature**: The initial temperature is chosen such that the corresponding acceptance ratio (i.e. the percentage of the proposed new states that are accepted) is $\chi_0 \approx 0.3$ [70],
- **Stop criterion**: The final energy is the same for a number (M) of consecutive temperatures,

- **Inner loop criterion**: The loop is executed a constant number of times (N),
- **Update of temperature:** $T := \lambda T$, where $\lambda \geq 0.99$.

Some experience is required to tune in appropriate values for χ_0, M, N, and λ. All the time a trade-off has to be made between a too fast cooling with the risk of a poor final solution, and a too slow cooling that is unnecessarily time-consuming. For a more extensive treatment of these parameters the reader is referred to the books [1, 44].

3 Construction Methods

This section contains the main results of our work. The construction methods have been divided into three groups: direct constructions, constructions of new codes from old, and finally a method which we have called "the matrix method" is treated. The reader should not forget that this is the author's classification, and the borderlines between the methods are in some cases vague. Before the presentation of the methods, we give two general rules.

Consider the introductory example in Section 1. Obviously, the order of the questions is not essential when determining the minimal set of answers needed.

Theorem 2 $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = K_{q_{i_1},q_{i_2},\ldots,q_{i_m}}(n_{i_1}, n_{i_2}, \ldots, n_{i_m}; R)$*, where* $i_1, i_2, \ldots, i_m$ *is a permutation of* $1, 2, \ldots, m$.

Proof: Assume that a code C_1 covers a space G_1 with covering radius R. Further assume that a space G_2 is obtained by permuting the coordinates of G_1. Now a code C_2 that covers G_2 with covering radius R can be constructed from C_1 by performing the same permutations on each codeword. □

A code that is obtained by permuting the coordinates of another code is said to be *equivalent* to the original code.

Secondly, in the definition of covering codes, we allow the arities of the spaces to be one. However, the following can be said when there are such *unary* coordinates.

Theorem 3 $K_{1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = K_{q_2,q_3,\ldots,q_m}(n_2, n_3, \ldots, n_m; R)$.

Proof: Let C_1 be a code that covers $F_{q_2}^{n_2} F_{q_3}^{n_3} \cdots F_{q_m}^{n_m}$ with covering radius R. Then the extended code C_2 obtained by adding n_1 zeroes to the beginning of each codeword in C_1 clearly covers $F_1^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$ with covering radius R, since any words and codewords have the same values in the first n_1 coordinates. Similarly, puncturing n_1 times a code that covers $F_1^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$ with covering radius R, gives a code that covers $F_{q_2}^{n_2} F_{q_3}^{n_3} \cdots F_{q_m}^{n_m}$ with covering radius R. This completes the proof. □

3.1 Direct Constructions

If we do not use old codes in a construction, we call it a *direct construction*. Actually, the last method of this section (the matrix method) is also such a method, but due to its special characteristics we have given it a subsection of its own.

3.1.1 General Cases

For some parameters it is possible to give explicit general constructions for mixed covering codes of minimal cardinality. For $R = 0$ we get the following theorem.

Theorem 4 $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; 0) = \prod_{i=1}^{m} q_i^{n_i}$.

Proof: If $R = 0$, then a covering code C must contain all words in the space, since if there is a word $x \notin C$, then for all $c \in C$, $c \neq x$, so for all $c \in C$, $d(c, x) \geq 1$. One occurrence of a codeword in the code is sufficient, so the cardinality of a minimal code equals the cardinality of the space, i.e., $\prod_{i=1}^{m} q_i^{n_i}$. □

The first part of [14, Theorem 14] is easily generalizable to the mixed case. In the following we let $n = \sum_{i=1}^{m} n_i$ (> 0).

Theorem 5 *If* $n \leq R$, *then* $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = 1$.

Proof: If the length of the code is $n \leq R$, then any single word in the space is a covering code, since any word can differ from this maximally in $n \leq R$ coordinates. As we do not allow empty sets of codewords, it follows that $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = 1$. □

Short codes can be explicitly constructed. Minimal covering codes of length 1 and 2 are completely determined by Theorems 4 and 5, except for the case $n = 2$, $R = 1$. To deal with this case, we generalize [38, Lemma 1]. We first permute the coordinates so that $q_1 \leq q_2 \leq \cdots \leq q_m$ ($\lceil x \rceil$ ($\lfloor x \rfloor$) stands for the smallest (greatest) integer greater (less) than or equal to x).

Theorem 6 *If* $q_1 \leq q_2 \leq \cdots \leq q_m$ *and* $R + 1 \leq n \leq R + \lceil n_1/q_1 \rceil$, *then* $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = q_1$ $(n_1 \geq 1)$.

Proof: $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) \leq K_{q_1}(n_1, R - (n - n_1)) = q_1$ by [10, Theorem 6], so q_1 is an upper bound.

On the other hand, for any code C with $q_1 - 1$ codewords there is always one word u such that $d(u, C) = n > R$. Thus $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = q_1$. □

In [37], Theorem 6 can for example be used to prove that $K_{3,2}(1, 5; 3) = 2$. For $n_1 = 1$, we obtain the following result.

Corollary 1 *If* $q_1 \leq q_2 \leq \cdots \leq q_m$ *and* $n = R + 1$, *then* $K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R) = q_1$ $(n_1 \geq 1)$.

3.1.2 Codes of Length 3

Now, moving on to length 3, minimal cardinalities are given by the theorems presented, except for the case $n = 3$, $R = 1$. Again, we generalize a method presented in [38] to obtain an upper bound for these codes. The idea behind the result in [38] (that treats nonmixed codes) is to partition the set of possible values of the coordinates into two sets, e.g., $F_3 = \{0, 1, 2\} = \{0\} \cup \{1, 2\}$. Then, for any word, at least two of the coordinates will have values that belong to one of these sets. Thus, for each of these two sets we have to construct codewords, such that for any two coordinates any pair of values (belonging to the set in question) will occur in some codeword. If a set has q values, then the number of codewords constructed with values from this set must apparently be at least q^2. As a matter of fact, there are always q^2 such codewords. Then, e.g., if $q = 3$, $K_3(3, 1) \leq 1^2 + 2^2 = 5$. The following general theorem [38, Theorem 1] can be proved.

Theorem 7 $K_q(3, 1) = \lfloor (q^2 + 1)/2 \rfloor$.

Proof: See proof of [38, Theorem 1]. □

Before presenting a result for the mixed case, we prove the following lemma.

Lemma 1 *If $q'_1 \leq \min\{q'_2, q'_3\}$, then there is a code $C \subseteq F_{q'_1} F_{q'_2} F_{q'_3}$ with $q'_2 q'_3$ codewords, such that for any two coordinates, any pair of values occurs in some codeword.*

Proof: We show that the following code $C = \cup(a_i, b_i, c_i)$ fulfills the requirements. Take as codewords all $q'_2 q'_3$ words such that all possible pairs of values in the last two coordinates occur in some word, and let $a_i \equiv b_i + c_i \pmod{q'_1}$. For a fixed c_i, as b_i goes through all its values cyclically, so does a_i, thus (since $q'_1 \leq q'_2$) all possible pairs (a_i, c_i) occur. Similarly all pairs (a_i, b_i) occur. □

Actually, it is easily seen that there is no code with fewer than $q'_2 q'_3$ codewords that has the same property, since as any pair must occur, the number of codewords must be at least $\max\{q'_1 q'_2, q'_1 q'_3, q'_2 q'_3\} = q'_2 q'_3$.

Theorem 8 *If $q_1 \leq q_2 \leq q_3$ and $q_1 < q_3$, then $K_{q_1,q_2,q_3}(1, 1, 1; 1) \leq \lfloor q_2 q_3 / 2 \rfloor$.*

Proof: Also in the case of mixed codes, we partition the sets of values of the coordinates into pairs of sets. That is, $F_{q_1} = F'_1 \cup F''_1 = \{0, 1, \ldots, q'_1 - 1\} \cup \{q'_1, q'_1 + 1, \ldots, q_1 - 1\}$, etc. The cardinalities of these sets satisfy the following equations: $q'_1 + q''_1 = q_1$, $q'_2 + q''_2 = q_2$, $q'_3 + q''_3 = q_3$. Now, for any word, at least two of the coordinates, i and j, will have values that both belong to F'_i and F'_j

or to F_i'' and F_j''. The construction in Lemma 1 can then be used to construct codes C_1' and C_2', such that both these values occur in some codeword.

Let $q_1' = \lfloor q_1/2 \rfloor$, $q_1'' = \lceil q_1/2 \rceil$, $q_2' = \lfloor q_2/2 \rfloor$, $q_2'' = \lceil q_2/2 \rceil$, $q_3' = \lceil q_3/2 \rceil$, and $q_3'' = \lfloor q_3/2 \rfloor$. Obviously $q_1' < \min\{q_2', q_3'\}$ and $q_1'' \leq \min\{q_2'', q_3''\}$ (since $q_1 < q_3$). Using Lemma 1 we can now construct a code with $M = \lfloor q_2/2 \rfloor \cdot \lceil q_3/2 \rceil + \lceil q_2/2 \rceil \cdot \lfloor q_3/2 \rfloor$ codewords and covering radius 1. Simplifying this expression we get that the number of codewords is $M = \lfloor q_2 q_3/2 \rfloor$. □

Using this result we can prove the following bounds in [37]: $K_{3,2}(2,1;1) \leq 4$, $K_{3,2}(1,2;1) \leq 3$.

It is essential that we do not allow all the cardinalities to be equal. If $q_2 \leq q_3$ are both odd, then if $q_1 = q_2 = q_3$, $K_{q_1,q_2,q_3}(1,1,1;1) = (q_2 q_3 + 1)/2$ (Theorem 7), but if $q_1 < q_3$, then $K_{q_1,q_2,q_3}(1,1,1;1) \leq (q_2 q_3 - 1)/2 < (q_2 q_3 + 1)/2$.

The bound is not exact (cf. Theorem 7). Another upper bound is for example $q_1 q_2$, obtained by constructing codewords with all possible pairs of (all) values in the first two coordinates. For $q_3 \geq 2q_1 + 1$, this bound is better than the bound obtained using Theorem 8. Using the following recursive formula (implying that it is not actually a direct construction) together with Theorem 8 enables us to calculate good upper bounds for most values of q_1, q_2 and q_3.

Theorem 9 $K_{q_1,q_2,\ldots,q_n}(1,1,\ldots,1;n-2) \leq q_1 + K_{q_1,q_2-1,q_3-1,\ldots,q_n-1}(1,1,\ldots,1;n-2)$.

Proof: Let C' be a $(q_1, q_2-1, q_3-1, \ldots, q_n-1; 1,1,\ldots,1; M')n-2$ code. We now construct $C = C' \cup C''$, where $C'' = F_{q_1} \oplus (q_2-1, q_3-1, \ldots, q_n-1)$. This code has $q_1 + M'$ codewords, so it remains to show that it has covering radius $n-2$. Take any $x = (x_1, x_2, \ldots, x_n) \in F_{q_1} F_{q_2} \cdots F_{q_n}$. Now, if $x_i = q_i - 1$ for any $i \in \{2, 3, \ldots, n\}$, then $d(x, C) \leq d(x, C'') \leq n-2$. Otherwise, $x \in F_{q_1} F_{q_2-1} \cdots F_{q_n-1}$, but then $d(x,C) \leq d(x,C') \leq n-2$, so in any case $d(x,C) \leq n-2$. □

Observe that we do *not* require the arities to be in order of magnitude, however, the best bounds are obtained when q_1 is the smallest of them. Now, e.g., $K_{3,4,5}(1,1,1;1) \leq 3 + K_{3,3,4}(1,1,1;1) \leq 3 + \lfloor 3 \cdot 4/2 \rfloor = 9$, which is better than the bound obtained using Theorem 8 with the original parameters ($\lfloor 4 \cdot 5/2 \rfloor = 10$).

Using simulated annealing (to be discussed further on) we have checked whether we could find better bounds than those given by Theorems 8 and 9. The bounds obtained for $q_1, q_2, q_3 \leq 5$ are displayed in Table 1.

q_1	q_2	q_3	$K_{q_1,q_2,q_3}(1,1,1;1)$	q_1	q_2	q_3	$K_{q_1,q_2,q_3}(1,1,1;1)$
2	2	2	2	3	3	3	5
2	2	3	3	3	3	4	6
2	2	4	4	3	3	5	7
2	2	5	4	3	4	4	8
2	3	3	4	3	4	5	9
2	3	4	5	3	5	5	10
2	3	5	6	4	4	4	8
2	4	4	6	4	4	5	10
2	4	5	7	4	5	5	12
2	5	5	8	5	5	5	13

Table 1: Upper bounds for mixed codes with $n = 3$, $R = 1$.

One improvement could be made, viz. $K_{3,5}(1,2;1) \leq 10$. This bound is proved by the following code: $C = C_1 \cup C_2 \cup C_3$, where $C_1 = \{000, 100\}$, $C_2 = \{021, 012, 222, 211\}$, and $C_3 = \{243, 234, 144, 133\}$. The code is presented partitioned in order to reveal the structure of it. The construction is similar to that in the proof of Theorem 9. A slight improvement can be made since C_1 does not have to contain the word 200. This is due to the fact that for any $a \in \{1,2,3,4\}$ there are codewords $(2,a,x)$ and $(2,x,a)$ in $C_2 \cup C_3$.

3.1.3 Codes of Length at Least 4

In [60] the cases $n = 4, 5$, $R = 1$ are also considered. Unfortunately, we have not been able to give a nice generalization for these parameters in case of mixed covering codes. For $n \geq 4$, $R = n - 2$, we can use the same method as for $n = 3$, $R = 1$ to construct covering codes.

A $(q_1, q_2, \ldots, q_n; 1, 1, \ldots, 1; M)R$ code C is called *s-surjective* if for any set of s coordinates $a_1, a_2, \ldots, a_s$ of C and any s-tuple $(b_1, b_2, \ldots, b_s) \in F_{q_{a_1}} F_{q_{a_2}} \cdots F_{q_{a_s}}$ there is a codeword $(c_1, c_2, \ldots, c_n)$ such that for $1 \leq i \leq s$, $c_{a_i} = b_i$. In the nonmixed case, s-surjectivity has been treated, e.g., in [34, 57]; see also the references mentioned in those papers. The first discussion on s-surjectivity and covering codes appeared in [31]. An interesting function is $\mathrm{ms}_q(n,s)$ ($s_q(n,s)$ is used in [57]), denoting the smallest possible number of codewords in an s-surjective q-ary code of length n. In the mixed case we use $\mathrm{ms}_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; s)$. In Lemma 1 and the discussion subsequent to that lemma we proved the following.

Theorem 10 *If* $q_1 \le q_2 \le q_3$*, then* $ms_{q_1,q_2,q_3}(1,1,1;2) = q_2 q_3$.

Assume that $q_1 \ge q_2 \ge \cdots \ge q_n$. It is easily seen that $\mathrm{ms}_{q_1,q_2,\ldots,q_n}(1,1,\ldots,1;1) = q_1$ and that $\mathrm{ms}_{q_1,q_2,\ldots,q_n}(1,1,\ldots,1;s) \ge \prod_{i=1}^{s} q_i$. In the nonmixed case this bound is exact for $s=2$ if and only if there exist $n-2$ mutually orthogonal Latin squares of order q, see [3, Theorem 5.2.1]. (For the first connection between Latin squares and covering codes, see [39]) The maximal number of mutual orthogonal Latin squares of order q is denoted by $N(q)$. If q is a prime power, then $N(q) = q-1$ [3, Theorem 5.1.1]. It is further known that $N(6) = 1$ and that $N(q) \ge 2$, for $q > 6$. In the binary case we know the exact values for $\mathrm{ms}_2(n,2)$ (see [2, Corollary 5.2.4] and [43]).

Theorem 11 $ms_2(n,2) = \min_M n \le \binom{M-1}{\lfloor (M-2)/2 \rfloor}$.

We now prove the following general theorem.

Theorem 12 $K_{q_1,q_2,\ldots,q_n}(1,1,\ldots,1;n-2) \le \sum_{i=1}^{n-1} ms_{q_{i,1},q_{i,2},\ldots,q_{i,n}}(1,1,\ldots,1;2)$*, where for all* j, $\sum_{i=1}^{n-1} q_{i,j} = q_j$.

Proof: For any $n > 3$, the sets of possible values of the coordinates are partitioned into $n-1$ subsets. Now for any $x \in H$, at least two of the values of the coordinates must belong to the same subset. The code can now be constructed so that for any two coordinates any pair of values from the same subset will occur in some codeword, that is, the code is a union of $n-1$ 2-surjective subcodes, and the theorem follows. □

Now, we get the following sequence of bounds for $n=4$: ($\mathrm{ms}_2(4,2) = 5$, $\mathrm{ms}_q(4,2) = q^2$ for $q \ne 2,6$) $K_q(4,2) \le 3,7,11,15,19,23,27$ for $q = 3,4,5,6,7,8,$ and 9, respectively. (This is not an arithmetic series!) Thus, we have an alternative proof for $K_5(4,2) \le 11$, which was proved in [69] by explicitly listing the codewords of a corresponding covering code. The bound $K_4(4,2) \le 7$ has been proved to be exact by Honkala [10]. An incorrect value (8) for these parameters has been reported in [7] (the notation used in that paper is $\gamma(n,k,s) = K_k(n,n-s)$). Another incorrect value in the same paper is $K_3(5,2) = 9$, it is currently known that $6 \le K_3(5,2) \le 8$ [69].

For $n=5$, some values of $\mathrm{ms}_q(5,2)$ are unknown, the most interesting $\mathrm{ms}_3(5,2)$ (> 9). The 2-surjective code in Figure 6 proves that $\mathrm{ms}_3(5,2) \le 11$.

For $n=5$ the following bounds are then obtained: $K_q(5,3) \le 4,9,14,19,24,29$ for $q = 4,5,6,7,8,$ and 9, respectively. $K_5(5,3) \le 9$ is as a matter of fact a new record! The previous best known upper bound (10) was also obtained by a combinatorial construction [69].

00000	10111	21012	11220
01121	20222	11102	22110
02212	12021	22201.	

Figure 6: Ternary 2-surjective code of length 5.

For lower bounds the reader is referred to [56], where it is shown that $K_q(n, n-2) \geq q^2/(n-1)$, and that this bound is exact if and only if $(n-1)$ divides q, and $N(q/(n-1)) \geq n-2$ [56]. Thus, e.g. $K_9(4,2) = 27$.

Finally, we have used simulated annealing to find upper bounds for the case $n = 4$, $R = 1$. In this case exact values are known only for nonmixed code with $q = 2, 3$, and 4. The upper bounds are presented in Table 2.

We do not further analyze these results here; we just notice that the construction methods to be presented in the subsequent subsection can be used to explain many of the bounds. Further analysis of bounds that cannot be easily explained might possibly lead to new construction methods. This is one of the nice properties of simulated annealing. Although the results obtained might not be of very big general interest, new insight might be obtained by deep examination of these. This has been pointed out earlier by van Lint in [50].

3.1.4 Constructions of Perfect Mixed Covering Codes

Perfect codes (we omit the word covering in the sequel, since perfect covering codes are also perfect packing codes) are interesting, since the codewords of such codes are uniformly distributed over the space, and the number of codewords in the code attains the theoretical minimum given by Theorem 1. They can thus be considered very *efficient* codes. Examples of *trivial* perfect codes are $C = H$ in the case $R = 0$, and $C = \{x\}$, where $x \in H$ is arbitrary, in the case $R = n$ (n is the length of the code). In the nonmixed case, the area of perfect codes have been quite exhaustively studied especially by van Lint and Tietäväinen, see [48] for an early survey.

Construction methods for perfect mixed codes have been presented, mainly in [25, 26, 27, 47]. In these papers it is showed that the existence of such codes is closely related to the existence of certain partitions of finite Abelian

q_1	q_2	q_3	q_4	$K_{q_1,\ldots,q_4}(1,\ldots;1)$	q_1	q_2	q_3	q_4	$K_{q_1,\ldots,q_4}(1,\ldots;1)$
2	2	2	2	4	2	4	5	5	22
2	2	2	3	6	2	5	5	5	26
2	2	2	4	7	3	3	3	3	9
2	2	2	5	8	3	3	3	4	12
2	2	3	3	6	3	3	3	5	15
2	2	3	4	8	3	3	4	4	16
2	2	3	5	10	3	3	4	5	20
2	2	4	4	8	3	3	5	5	24
2	2	4	5	10	3	4	4	4	21
2	2	5	5	12	3	4	4	5	25
2	3	3	3	9	3	4	5	5	32
2	3	3	4	12	3	5	5	5	37
2	3	3	5	13	4	4	4	4	24
2	3	4	4	12	4	4	4	5	30
2	3	4	5	15	4	4	5	5	38
2	3	5	5	17	4	5	5	5	42
2	4	4	4	16	5	5	5	5	51
2	4	4	5	20					

Table 2: Upper bounds for mixed codes with $n = 4$, $R = 1$.

groups. Among the classes of perfect mixed codes that have been proved to exist, we here show the following result.

Theorem 13 *If q is prime power, then there exists a perfect covering of $F_{q^a}F_{q^b}^{n-1}$ $(a \geq b)$ with covering radius one if and only if there is an $r \geq a+b$, such that*

$$q^r = q^a + (n-1)(q^b - 1).$$

Proof: Follows from [47, Theorem 1] and [47, Theorem 2]. □

If some of the coordinates are binary, then necessarily $q = 2$ and $b = 1$, giving the following corollary.

Corollary 2 *For $m > a \geq 2$, there is a perfect covering of $F_{2^a}F_2^{n-1}$ $(a \geq 1)$ with covering radius one if and only if $n = 2^m - 2^a + 1$.*

It was for a long time unknown whether perfect mixed codes exist for $R > 1$. However, such codes have now been found by Etzion and Greenberg [19]. The construction is based on the partition of the extended Hamming code into disjoint translates of the Preparata code (see [55] for details about these concepts of coding theory).

Theorem 14 *For $n \geq 4$ even, $K_{2^{n-1},2}(1, 2^n; 2) = 2^{2^n-n-1}$.*

Proof: See proof of [19, Theorem 2]. □

3.1.5 Using SA in the Construction of Covering Codes

Constructing covering codes by hand is generally possible only as for codes with up to 5–6 codewords; exhaustive computer searches can manage somewhat larger codes. In any case, the number of inequivalent codes grows extremely fast as the length and the number of codewords increase. One possibility to deal with this problem is to use some kind of probabilistic methods.

Simulated annealing is such a method. The first attempts to use SA in the construction of codes were made by El Gamal *et al.* [18]. However, in that paper covering codes are not treated. The first results concerning such codes were presented by Wille [67]. The results turned out to be very promising, and SA has been used to construct many record-breaking covering codes during the last few years [45, 68, 69].

The implementation of the direct approach is quite straightforward. The energy function is simply the number of words not covered by the codewords.

When we are searching for a code C that covers H with radius R the energy function is

$$E = |\{x \in H \mid d(x, C) > R\}|.$$

For covering codes, the value of this function is apparently 0. In the annealing process we consider the codewords one at a time. A neighbourhood configuration is obtained by replacing a codeword c with a randomly chosen word c', such that $1 \leq d(c, c') \leq R$. We could also for example choose c' in a completely random way, but the previous condition has turned out to lead to the best performance. The codeword to be replaced is not either chosen in a random way, but all codewords are considered in turn.

The bigger the radius and the number of codewords to find are, the more time-consuming the SA algorithm becomes. Finally, a point where the algorithm is not feasible any more is reached, and other methods have to be considered. The possibility of using old codes to construct new ones is often utilized in coding theory.

3.2 Constructing New Codes from Old

3.2.1 Simple Constructions

A classic method to combine codes is the *direct sum construction* [55, p. 76]. The direct sum of two codes $C_1 = (q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M_1)R_1$ and $C_2 = (q_1, q_2, \ldots, q_m; n'_1, n'_2, \ldots, n'_m; M_1)R_2$ is $C_1 \oplus C_2 = (q_1, q_2, \ldots, q_m; n_1 + n'_1, n_2 + n'_2, \ldots, n_m + n'_m; M_1 M_2)R_1 + R_2$.

Theorem 15 *$K_{q_1,q_2,\ldots,q_m}(n_1 + 1, n_2, \ldots, n_m; R) \leq q_1 \cdot K_{q_1,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R)$.*

Proof: Let C_1 be a code with the previously mentioned parameters, and let C_2 be the $(q_1, 1, q_1)0$ code consisting of all words in the space F_{q_1}. Then $C_1 \oplus C_2 = (q_1, q_2, \ldots, q_m; n_1 + 1, n_2, n_3 \ldots, n_m; q_1 M_1)R_1$. □

Constructions (a) and (b) in [37] are special cases of this theorem. From Theorem 15 it is clear why we accept $n_i = 0$ in the definition of mixed covering codes. The generalization of constructions (c) and (d) in [37] is as follows.

Theorem 16 *$K_{q_1,q_2,\ldots,q_m}(n_1, n_2 + 1, \ldots, n_m; R) \leq$*

1. $K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)+(q_2-q_1)\cdot\lfloor K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)/q_1\rfloor$, *if* $q_2>q_1$,
2. $K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)$, *if* $q_2\le q_1$.

Proof: (1) Assume that C is a $(q_1,q_2,\ldots,q_m;n_1+1,n_2,\ldots,n_m;M)R$ code of length n. As in [37] we define $C'_a=C\cup\{(b,x_2,\ldots,x_n)\in F_{q_1}^{n_1}F_{q_2}^{n_2}\cdots F_{q_m}^{n_m}\mid (a,x_2,\ldots,x_n)\in C, b\in\{q_1,\ldots,q_2-1\}\}$ for $a\in F_{q_1}$. For all these a the $(q_1,q_2,\ldots,q_m;n_1,n_2+1,\ldots,n_m;M_a)R_a$ codes C'_a satisfy $R'_a=R$. There is further an $i\in F_{q_1}$ such that the number of codewords in C with value i in the first coordinate is less than or equal to $\lfloor|C|/q_1\rfloor$. Then $|C'_i|\le|C|+(q_2-q_1)\cdot\lfloor|C|/q_1\rfloor$.

(2) The second part of the theorem is proved by changing the values of the first coordinate of C, so that values in the interval $q_2,\ldots,q_1$ are changed to 0's (or any other value in the interval $0,\ldots,q_2-1$). The covering radius is then $\le R$, and the code has become q_2-ary in the first coordinate. This completes the proof. □

We could also have used $q_2/q_1\cdot K_{q_1,q_2,\ldots,q_m}(n_1+1,n_2,\ldots,n_m;R)$ as an upper bound in the first part of Theorem 16, but the expression used gives tighter bounds in some cases. Certainly, an investigation of the values of the coordinates of a code C could give even better codes, in cases the distribution of the values is not even.

3.2.2 Partitioning Codes

Many good construction methods require that the codes have certain *partitioning properties*. The *amalgamated direct sum* (ADS) construction can, e.g., efficiently be applied to two *normal* codes. These concepts were introduced in [24] (the linear case) and [14] (the general case). *Subnormality* was introduced by Honkala in [29] and in [31] it was shown that the ADS construction can be used efficiently to combine a normal and a subnormal code.

In [54] Lobstein and van Wee generalize the concepts of normality and subnormality to the q-ary case. Their results indicate that these concepts are not as important in the general case as in the binary case. However, in [69], the author of this report presents another definition of normality for q-ary covering codes, named (k-)seminormality. This concept has turned out to be very useful in the construction of q-ary covering codes ($q\ge3$). Inspired by these results, Honkala has published a generalized theory in [32], where (k,t)-subnormal codes are defined. Finally, (k,t)-normal codes have been introduced in [71].

We here give the definitions of (k, t)-normal and (k, t)-subnormal mixed covering codes, following the approaches in [32, 33, 69, 71], which can also be consulted for further details. The definitions are essentially the same as those for nonmixed codes.

Subnormal Codes

Definition 5 *Suppose C is a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code. Then C has (k, t)-subnorm (S, R') if there is a partition of C into k nonempty subsets $C_0, C_1, \ldots, C_{k-1}$ such that*

$$\min_a d(x, C_a) + \max_a d(x, C_a) \leq S$$

whenever

$$R - t \leq d(x, C) \leq R,$$

and for all x

$$\max_a d(x, C_a) \leq R'.$$

Such a partition is called acceptable. *If the subnorm is given as a single value and not as tuple, it refers to the S parameter. If C has (k, t)-subnorm $2R+1$, then it is called (k, t)*-subnormal. *If C is not (k, t)-subnormal, then it is called (k, t)*-absubnormal. *If C is $(k, 0)$-subnormal, then it is called k*-seminormal. *If C is $(2R + 1, R + 1)$-subnormal, then it is called* strongly k-seminormal.

The following theorem (see [71, Theorem 1]) shows how codes can be combined using the *blockwise direct sum* (BDS), introduced by Honkala in [32].

Theorem 17 *Let A be a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M_A)R_A$ code, and let B be a $(q'_1, q'_2, \ldots, q'_{m'}; n'_1, n'_2, \ldots, n'_{m'}; M_B)R_B$ code. Assume further that A has (k, R_A)-subnorm (S_A, R'_A) with the partition $A = A_1 \cup \ldots \cup A_k$ acceptable, and that B has (k, t_B)-subnorm S_B with the partition $B = B_1 \cup \ldots \cup B_k$ acceptable. Then the* blockwise direct sum *of A and B*

$$BDS(A, B) = \bigcup_{i=1}^{k} (A_i \oplus B_i)$$

is a $(q_1,\dots,q_m,q'_1,\dots q'_{m'};n_1,\dots,n_m,n'_1,\dots,n'_{m'};M)R$ *code with* $M = \sum_{i=1}^{k}|A_i|\cdot|B_i|$ *and* $R \le \max\{R_1,R_2\}$, *where*

$$R_1 = \lfloor (S_A+S_B)/2 \rfloor$$
$$R_2 = \begin{cases} 0 & \text{if } t_B \ge R_B \\ R_B - t_B - 1 + R'_A & \text{otherwise.} \end{cases}$$

Proof: We prove that $\max\{R_1,R_2\}$ is an upper bound for the covering radius of BDS(A,B). First we denote $G = F_{q_1}^{n_1}F_{q_2}^{n_2}\cdots F_{q_m}^{n_m}$, and $G' = F_{q'_1}^{n'_1}F_{q'_2}^{n'_2}\cdots F_{q'_{m'}}^{n'_{m'}}$. Let $x=(u,v)\in GG'$, $u\in G$ and $v\in G'$.

We first consider the case $d(v,B)\ge R_B - t_B$. Let $d(u,A)=d(u,A_\alpha)=a$, $d(v,B)=d(v,B_\beta)=b$. By our assumption we now have $d(u,A_i)\le S_A - a$, $d(v,B_i)\le S_B - b$ for all i. Hence $2d(x,\text{BDS}(A,B)) \le d(x,A_\alpha\oplus B_\alpha)+d(x,A_\beta\oplus B_\beta) = d(u,A_\alpha)+d(v,B_\alpha)+d(u,A_\beta)+d(v,B_\beta) \le a+(S_B-b)+(S_A-a)+b = S_A+S_B$, so $d(x,\text{BDS}(A,B))\le \lfloor (S_A+S_B)/2\rfloor$.

If $d(v,B)\le R_B-t_B-1$, $R_B-t_B-1\ge 0$, then there is a β such that $d(v,B_\beta)\le R_B-t_B-1$, so $d(x,\text{BDS}(A,B))\le d(x,A_\beta\oplus B_\beta)=d(u,A_\beta)+d(v,B_\beta)\le R_B-t_B-1+R'_A$. This completes the proof. □

The theorem is stated in a very general form. By examining special cases, more practical forms can be derived. Among other, the following very useful theorem can be proved. (cf. [69, Theorem 3])

Theorem 18 *If there exists a q-seminormal* $(q_1,q_2,\dots,q_m;n_1,n_2,\dots,n_m;M)R$ *code, where q is a prime power, then* $K_{q,q_1,q_2,\dots,q_m}(q,n_1,n_2,\dots,n_m;R+1)\le q^{q-2}M$.

Proof: If q is a prime power, then there exists a $(q,q+1,q^{q-1})1$ Hamming code. Puncturing this code gives a $(q,q,q^{q-1})1$ code A, which has $(q,1)$-subnorm $(2,2)$ and there is an acceptable partition, where all subsets are of the same size (q^{q-2})[69, Theorem 2]. Further, let B be a $(q_1,q_2,\dots,q_m;n_1,n_2,\dots,n_m;M)R$ q-seminormal code, i.e., B has $(q,0)$-subnorm $2R+1$. Theorem 17 now gives that BDS(A,B) is a $(q,q_1,q_2,\dots,q_m;q,n_1,n_2,\dots,n_m;q^{q-2}M)R{+}1$ code, and the theorem follows. □

Again, a special case of this result is proved in the following result, first presented by Honkala in [30, Theorem 8].

Theorem 19 *If $n_1\ge 1$ and there is an $i\ge 2$ such that $n_i\ge 1$, then* $K_{2,q_2,\dots,q_m}(n_1+3,n_2,\dots,n_m;2)\le K_{2,q_2,\dots,q_m}(n_1+1,n_2,\dots,n_m;1)$.

Proof: From [30, Theorem 7] we know that every $(2, q_2, \ldots, q_m; n_1+1, n_2, \ldots, n_m; K_{2,q_2,\ldots,q_m}(n_1+1, n_2, \ldots, n_m; 1))1$ code of length at least 2 is 2-seminormal (the theorem is actually even stronger than that). Letting $q = 2$ in Theorem 18 gives the desired result. □

The theorem is closely related to a conjecture first presented by Cohen et. al. [13]. They conjectured that $K_2(n+2, R+1) \leq K_2(n, R)$ for $n \neq R$. It has been proved that this inequality holds in many special cases, but a complete proof is still missing. We extend the conjecture to concern mixed covering codes with at least one binary coordinate ($n = \sum_{i=1}^{m} n_i$).

Conjecture 1 *If $n_1 \geq 1$ and $n \neq R$, then $K_{2,q_2,\ldots,q_m}(n_1+2, n_2, \ldots, n_m; R+1) \leq K_{2,q_2,\ldots,q_m}(n_1, n_2, \ldots, n_m; R)$.*

Normal Codes Let C be a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code with $C_a^{(i)\prime} = \{(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \mid (c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) \in C\}, a \in F_q$, where F_q is the arity of the i^{th} coordinate. These sets are assumed to be nonempty for all a.

Definition 6 *The code C has (q,t)-norm $(N^{(i)}, R_i')$ with respect to the i^{th} coordinate if*

$$\min_a d(x, C_a^{(i)\prime}) + \max_a d(x, C_a^{(i)\prime}) \leq N^{(i)}$$

whenever

$$R - t \leq d(x, \bigcup_a C_a^{(i)\prime}) \leq R,$$

and for all x

$$\max_a d(x, C_a^{(i)\prime}) \leq R_i'.$$

If

$$N^{(i)} \leq N, R_i' \leq R'$$

*for at least one coordinate i, we say that C has (q,t)-*norm *(N, R') (or N alone, cf. the definition of (q,t)-subnorm), and the coordinates i for which this equation holds are called* acceptable. *If C has (q,t)-norm $2R$ it is called (q,t)-*normal. *If C is not (q,t)-normal it is called (q,t)-*abnormal.

If a code C has (q,t)-subnorm (N,R'') with respect to the i^{th} coordinate, then the deletion of this coordinate gives a code C' that has (q,t)-subnorm (N,R'') if C and C' have the same covering radius, otherwise C' has $(q,t-1)$-subnorm (N,R''). Namely, puncturing a code with covering radius R might lead to a code with covering radius $R-1$. The following can be said about the $(q,0)$-norm of covering codes.

Lemma 2 *Let C be any $(q_1,q_2,\ldots,q_n;1,1,\ldots,1;M)R$ code. Then C has $(q_i,0)$-norm $2R$ with respect to the i^{th} coordinate.*

Proof: For any i and x such that $d(x,\bigcup_a C_a^{(i)\prime})=R$, assume that there is an a such that $d(x,C_a^{(i)\prime})\geq R+1$. Then $d((x,a),C_a^{(i)\prime}\oplus\{a\})\geq R+1$ and for all $b\in F_{q_i}\setminus\{a\}$, $d((x,a),C_b^{(i)\prime}\oplus\{b\})\geq R+1$. This contradicts the fact that C has covering radius R. □

The lemma will be used in the proof of the following theorem, which has turned out to be useful in many constructions (see [69, Theorem 4] and [71, Theorem 3] for results concerning nonmixed codes).

Let A be a $(q_1,q_2,\ldots,q_m;n_1,n_2,\ldots,n_m;M_A)R_A$ code and let B be a strongly q_m-seminormal $(q_1',q_2',\ldots,q_{m'}';n_1',n_2',\ldots,n_{m'}';M_B)R_B$ code ($n_m>0$, $B=B_0\cup\cdots B_{q_m-1}$ is an acceptable partition). We then define the ADS of A and B as

$$C=A\dot{\oplus}B=\bigcup_{i=0}^{q_m-1}\{(a,b)\mid(a,i)\in A, b\in B_i\}.$$

Theorem 20 *$C=A\dot{\oplus}B$ is a $(q_1,\ldots,q_{m-1},q_m,q_1',\ldots q_{m'}';n_1,\ldots,n_{m-1},n_m-1,n_1',\ldots,n_{m'}';M)R_C$ code with $M=\sum_{i=0}^{q_m-1}|A_i^{(n)}|\cdot|B_i|$ codewords and $R_C\leq R_A+R_B$.*

Proof: The length and the number of elements of C is immediately clear ($A_i^{(n)}$ denotes the set of codewords that have value i in the n^{th} coordinate), so it remains to prove the claimed covering radius of C.

If the punctured code A' has covering radius R_A then B has $(q,0)$-subnorm $2R_B$ (Lemma 2 and the discussion after Definition 6). Then the result follows from Theorem 17, as B has (q,R_B)-subnorm $(2R_B+1,R_B+1)$.

If A' has covering radius R_A-1, then $R_C\leq(R_A-1)+(R_B+1)=R_A+R_B$, since B is strongly seminormal. □

As mentioned in [71], there is a strongly 4-seminormal $(2,3,8)0$ code, whose acceptable partition consists of the 4 cosets of the perfect $[3,1]1$ Hamming code over F_2. Generally, the following result can be obtained.

Lemma 3 *If q is a prime power and $k \geq 2$, then there exists a strongly q^k-seminormal $(q, (q^k-1)/(q-1), q^{(q^k-1)/(q-1)})0$ code.*

Proof: A space F_q^n (q is a prime power, $n = (q^k-1)/(q-1)$, $k \geq 2$) can be partitioned into the q^k cosets of the perfect $[n, n-k]1$ Hamming code over F_q. □

The following theorem generalizing [68, Theorem 1] can now be obtained.

Theorem 21 *If q is a prime power and $k \geq 2$, then $K_{q_1=q, q_2=q^k, \ldots, q_m}(n_1 + (q^k-1)/(q-1), n_2, \ldots, n_m; R) \leq q^{(q^k-1)/(q-1)-k} \cdot K_{q_1=q, q_2=q^k, \ldots, q_m}(n_1, n_2 + 1, \ldots, n_m; R)$.*

Proof: Follows from Theorem 20 and Lemma 3. □

Actually, Heden presented a similar construction in [25], but he restricted his discussion to the construction of perfect codes of covering radius one starting from other (mixed and nonmixed) perfect codes of covering radius one.

[68, Theorem 2] is simply obtained by applying Theorem 21 several times. Theorems 16 and 20 can be used to prove construction (e) in [37]. One ternary coordinate of the original code is first made quaternary (with unchanged R). The ADS of this code and the aforementioned strongly 4-seminormal $(2, 3, 8)0$ code concludes the construction. Construction (f) and (i) in the same paper can be seen as "going the other way" in Theorem 21. The original code in this construction consists of codewords that are direct sums of shorter words and cosets of the $[3, 1]1$ Hamming code over F_2. These cosets are now replaced by a coordinate with values in F_4. Finally, in (f) this coordinate is made ternary (Theorem 16), and in (i) the ADS of this code and a strongly 4-seminormal $(3, 4, 12)1$ is taken.

Theorem 21 can be used to explain the results by Fernandes and Rechtschaffen in [20]. They proved that there are 25 and 63 codewords covering $F_9F_3^3$ and $F_9F_3^4$, respectively, with covering radius 1. This gives $K_3(7, 1) \leq 225$ and $K_3(8, 1) \leq 567$. Significantly better upper bounds for $K_3(7, 1)$ and $K_3(8, 1)$ (186 and 486) have, however, recently been found using other methods [45].

Theorem 22 $K_q(3(q+1), 1) \leq q^{3(q-1)} \lfloor (q^4+1)/2 \rfloor$, *for q a power of a prime.*

Proof: From Theorem 7 we know that $K_q(3, 1) = \lfloor (q^2+1)/2 \rfloor$, so we get that $K_{q^2}(3, 1) = \lfloor (q^4+1)/2 \rfloor$. The desired result now follows by applying Theorem 21 three times. □

Corollary 3 $K_3(12, 1) \leq 29889$.

This corollary gives an upper bound for the football pool problem for 12 matches. The bound is first mentioned by Hämäläinen and Rankinen in [37], the idea of the construction was, however, presented in a Finnish football pool magazine as early as in 1949.

The family of perfect mixed codes obtained in Theorem 14 can be used to construct good binary codes. This construction was also presented in [19, Construction B].

Theorem 23 *For $n \geq 4$ even,* $K_2(2^n + 2^{n-1} - 1, 2) = 2^{2^n + 2^{n-1} - 2n - 1}$.

Proof: Follows when Theorem 21 is applied once to Theorem 14. □

Further results on seminormal and subnormal codes, and constructions where such codes play a central role can be found in [32, 69, 71]. The applications of the constructions to mixed covering codes are also discussed in these papers.

Using SA in the Partitioning of Codes When we use codes in the the constructions that have been presented in this subsubsection we need to know their (k, t)-subnorm (and sometimes (k, t)-norm) and the corresponding acceptable partitions. The calculation of the (k, t)-norm is straightforward (cf. Definition 6).

The determination of a (good) (k, t)-subnorm (and an acceptable partition) for a code, however, is a much more difficult task. For some codes it is possible to find out subnormality by a theoretical proof. It has for example been shown that all nontrivial linear q-ary codes with covering radius 1 (except for the ternary $[4, 2]1$ Hamming code) are $(q, 1)$-subnormal [33]. The partitioning of nonmixed codes using SA was first discussed in [71].

We now show how to prove that a $(q_1, q_2, \ldots, q_m; n_1, n_2, \ldots, n_m; M)R$ code has $(k, 0)$-subnorm S (the most intersting case is $S = 2R + 1$, when the code is $(k, 0)$-subnormal, i.e., k-seminormal) or (k, R)-subnorm (S', R'), where S' can be of any value. In the latter case we are interested in $R' = R + 1$, when the code is strongly seminormal. This follows from the fact that $\min_a d(x, C_a) + \max_a d(x, C_a) \leq R + (R + 1) = 2R + 1$, so if a code has (k, R)-subnorm $(S', R + 1)$, then it has (k, R)-subnorm $(2R + 1, R + 1)$.

The algorithm to be presented cannot efficiently be used to prove that a code has (k, t)-subnorm S for $t > 0$. However, it seems to be very rare for a code to have $(k, 0)$-subnorm S' but not to have (k, t)-subnorm S' for $t > 0$, especially in the case of optimal codes. (Such a code is for example the $(3, 6, 3)4$ repetition code, which has $(3, 0)$-subnorm 8, but has $(3, 1)$-subnorm S only for $S \geq 9$.) We can thus first determine a $(k, 0)$-subnorm and thereafter check whether the (k,t)-subnorm is the same (with respect to the same partition of the codewords as the acceptable partition for the $(k,0)$-subnorm).

Let C be a $(q_1,q_2,\dots,q_m;n_1,n_2,\dots,n_m;M)R$ code. For a fixed S, the following approach can be used to prove that C has $(k,0)$-subnorm S. We divide the M codewords (preferably randomly) into k sets. If k divides M, the size of the sets is M/k, otherwise the size of some sets is $\lfloor M/k \rfloor$ and of some sets $\lceil M/k \rceil$. These sets are named C_a, $a \in F_k$. Let

$$E = |\{(x,a) \in (H,F_k) \mid d(x,C_a) > S-R, d(x,C) = R\}|. \tag{1}$$

Apparently, an acceptable partition has been found, when $E=0$. The problem has now been formulated as a combinatorial optimization problem, where (1) is the function that we try to minimize. In the annealing process we consider all the codewords of all the sets, one at a time, and the random displacement consists of swapping the codeword in question and a randomly chosen codeword *in another set*. The fact that our attention is focused on words at distance R from the code enables very efficient computer implementations.

When we want to prove that C has (k,R)-subnorm (S,R'), we use the function

$$E = |\{(x,a) \in (H,F_k) \mid d(x,C_a) > R'\}|. \tag{2}$$

For an acceptable partition $E=0$. If the approaches to find acceptable partitions are unsuccessful, it is worth trying to partition other nonequivalent codes with the same parameters.

3.2.3 Obtaining Coverings from Good Near-Coverings

The method to be discussed here is not widely applicable, but gives interesting and record-breaking codes in some special cases. Consider an arbitrary code C with covering radius R. Assume that the first coordinate is q-ary. We then obtain $C' = \bigcup_{a\in F_q} C'_a = \bigcup_{a \in F_q}\{c \mid (a,c) \in C\}$ by puncturing it once. Let us now have a look at C'_0. It is easily seen that those words that are at a distance bigger than R from C'_0 must be within distance $R-1$ from one of $C'_1, C'_2, \dots, C'_{q-1}$.

The simplest case is $R=1$, $q=2$. Then C'_1 must contain all words w, such that $d(w,C'_0) > 1$. We can now simply try to find the best possible covering C'_0, let $C'_1 = \{w | d(w,C'_0) > 1\}$, and check whether the extended code $C = \{(a,c) | c \in C'_a\}$ has covering radius 1.

There is a code consisting of 36 codewords that covers $F_3^5F_2\backslash Q$ ($|Q| = 60$) with radius 1 ($K_{3,2}(5,1;1) \leq 54$, so the near-covering is very good). By proceeding according to the previous discussion, the code that is obtained turns out to cover $F_3^5F_2^2$ with radius 1. (We here talk about *the* code, which may not be correct. However, all attempts (using SA) to find this near-covering has ended up with the same code (up to equivalence), so we conjecture that there is only one such code.) As it has $36+60$ codewords, $K_{3,2}(5,2;1) \leq 96$, explaining the bound in [37]. From this code it is further possible (using Theorem 16 and the discussion after that) to construct codes proving that $K_{3,2}(6,1;1) \leq 132$ (best known) and $K_3(7,1) \leq 198$ (best known for short time in the late 1960's [36]).

3.2.4 A New Method

In the following theorem we are going to make use of the fact that appending a parity check bit to a $(2,n,m)R$ binary code C, and thereafter deleting *any* coordinate of the new code, gives a code, which has covering radius R (see, e.g., [4, Fact 2]).

Theorem 24 *If there is a $(2,n_1,M_1)R_1$ code C', a $(2,n_2,M_2)R_2$ code C'', and $R_1 \leq R_2-(n_2-n_1-2)$, then $K_{3,2}(n_2-n_1+1,n_1;R_2+1) \leq (M_1+M_2)$.*

Proof: We show that the code $C = (C' \oplus (2,2,\ldots,2)) \cup C'''$, where the all-2 vector is of length n_2-n_1+1 and C''' is obtained from C'' by adding an overall parity check bit, proves the upper bound.

Consider any word $u \in F_2^{n_1}F_3^{n_2-n_1+1}$. If the values of at least 2 of the last n_2-n_1+1 coordinates of u are 2, then $d(u,C) \leq R_1+(n_2-n_1+1-2) = R_1+(n_2-n_1-2)+1 \leq R_2+1$. Otherwise the value of at most one of the last n_2-n_1+1 coordinates is 2, so the values of these coordinates (except for at most one) belong to F_2. We then get that $d(u,C) \leq R_2+1$, since (according to the discussion preceding the theorem) the deletion of one of the last n_2-n_1+1 coordinates of C''' gives a $(2,n_2,M_2)R_2$ code. □

This principle has been used by many constructors of football pool systems. For example, let C' be a $(2,5,7)1$ code and C'' a $(2,7,16)1$ Hamming code. Theorem 24 now gives $K_{3,2}(3,5;2) \leq 23$, explaining the bound in [37].

3.3 The Matrix Method

The method to be presented in this subsection was first considered by Kamps and van Lint [41] and later generalized by Blokhuis and Lam [6]. Carnielli [8] and van Lint Jr. [51] have independently generalized it to arbitrary covering radii. We here show how it can be applied to the mixed case.

Let $A = (I; M) = (a_1, \dots, a_n)$ be an $r \times n$ matrix where I is the $r \times r$ identity matrix and M is a $r \times (n-r)$ matrix. The following restrictions has to be made to the A matrix. Two functions f_1 and f_2 associate the rows and the columns of A to arities, i.e. these are functions $f_1 : \{1, 2, \dots, r\} \to \{q_1, q_2, \dots, q_m\}$ and $f_2 : \{1, 2, \dots, n\} \to \{q_1, q_2, \dots, q_m\}$. Due to the I matrix part we have that for all $1 \le i \le r$,

$$a_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

and we require that $f_1(i) = f_2(i)$. For all $1 \le i \le r$ and $r + 1 \le j \le n$ we require that $a_{i,j} \in F_q$, if $f_1(i) = f_2(j)$ and that $a_{i,j} = 0$, if $f_1(i) \ne f_2(j)$.

For $s \in H = F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$ (so $f_1(1) = \cdots = f_1(n_1) = q_1$, etc.) the *$R$-covering of s using A*, $S_{A,R}(s)$, is defined as

$$S_{A,R}(s) = \{s + \sum_{j=1}^{n} \alpha_j a_j \mid \alpha_j \in f_2(j), |\{j \mid \alpha_j \ne 0\}| \le R, 1 \le j \le n\}.$$

Consequently, $A = I$ corresponds to covering in the traditional sense. A subset S of H *R-covers H using A* if

$$H = \bigcup_{s \in S} S_{A,R}(s).$$

For the following theorem we define a space $G = F_{f_2(1)} F_{f_2(2)} \cdots F_{f_2(n)}$, the length of which is n. In practise, the space can be expressed in a short form by grouping coordinates of equal arity.

Theorem 25 *If S R-covers H using an $r \times n$ matrix $A = (I; M)$, then $W = \{w \in G \mid Aw \in S\}$ R-covers G. $|W| = |S| \cdot \prod_{i=r+1}^{n} f_2(i)$.*

Proof: (Follows the proof of Theorem 2.1 in [6].) Take any $x \in G$. Then $Ax \in H$, so there are $s, \alpha_1, \dots, \alpha_R$ and $i_1, \dots, i_R$ such that $Ax = s + \sum_{j=1}^{R} \alpha_j a_{i_j}$. Then $A(x - \sum_{j=1}^{R} \alpha_j e_{i_j}) = s \in S$, where e_j is the jth unit vector in G. Hence $x - \sum_{j=1}^{R} \alpha_j e_{i_j} \in W$ and $d(x, W) \le R$.

To find W, we start by denoting an n-tuple in G by (u, v), dividing the word into the first r and the last $n - r$ components. Then $A(u; v)^T =$

$(I;M)(u;v)^T = u + Mv \in H$. For each $s \in S$, v can first be arbitrarily chosen and then we can choose $u = s - Mv$. We can choose v in $\prod_{i=r+1}^{n} f_2(i)$ ways, so $|W| = |S| \cdot \prod_{i=r+1}^{n} f_2(i)$. □

In the definitions we have had a very general approach, in the sense that we do not require coordinates of equal arity to be grouped, that is we could well have $H = F_2^2 F_3^3 F_2$. However, by requiring that, we get a matrix M that has an equivalent form as displayed in Figure 7. The values within an area A_i belong to F_{q_i}.

$$\begin{array}{|ccccc|}
\hline
\boxed{A_1} & & & & \\
 & \boxed{A_2} & & 0 & \\
 & & \ddots & & \\
 & 0 & & \boxed{A_{m-1}} & \\
 & & & & \boxed{A_m} \\
\hline
\end{array}.$$

Figure 7: Matrix form.

We have for example found 80 codewords (these can be obtained from the author on request) covering $F_2^5 F_3^3$ with radius 1 using

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.$$

Here $f_1(i) = 2$ for $1 \le i \le 5$, $f_1(i) = 3$ for $6 \le i \le 8$, $f_2(i) = 2$ for $1 \le i \le 5$, $i = 9, 10$, and $f_2(i) = 3$ for $6 \le i \le 8$, Theorem 25 then gives that there are $80 \cdot 2^2 = 320$ codewords covering $F_2^7 F_3^3$ with covering radius 1. Thus $K_{3,2}(3,7;1) \le 320$. This improves on the previous best known upper bound (332) [37].

Now SA can be used to find the codewords (S) and the matrix A (i.e. the M part), after first fixing the parameters of the spaces and $|S|$, r and n. The energy function is $E = |H \setminus \bigcup_{s\in S} S_{A,R}(s)|$ (cf. [45]). If the value of this function reaches 0, a solution is found. Since we now have to find both codewords in the set S, and the matrix M, it is not immediately clear how to use SA in the search for a covering. In [45] the approach is to change both the codewords in S and the matrix M during the annealing process. This method has turned out to perform well only when S is a small set.

Other methods to deal with this problem have been discussed in [70]. One of the most promising approaches seems to be the following. If there is a small number of nonequivalent matrices A, all these matrices can be considered in finding a covering set S. The number of nonequivalent matrices of size $r \times n$ over F_q having no repeated columns and no columns of zero is denoted by $\Phi_q(n,r)$ (cf. [59]). Tables on $\Phi_2(n,r)$ were first published by Slepian in [58], some values on the same function can also be found in [59]. Slepian uses $\bar{S}_{nr} = \Phi_2(n,r)$ in his paper. In Table 3 we present values of $\Phi_2(l+r,r)$ for small values of l and r.

$l\backslash r$	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8
2	-	1	4	8	14	22	32	44
3	-	1	5	15	38	80	151	266
4	-	1	6	29	105	312	821	1948
5	-	-	5	46	273	1285	5098	17934

Table 3: $\Phi_2(l+r,r), 1 \le l \le 5, 1 \le r \le 9$.

The task of producing $\Phi_q(n,r)$, $n > r$, inequivalent matrices having no repeated columns and no columns of zero is much more difficult than just enumerating them. Only for the simplest case (i.e. M has only one column) an easy explicit description of the classes can be given. $\Phi_q(n,n-1) = n-2$; M is then a single column vector with 2 to $n-1$ non-zero positions. This fact was used in the search of a covering code improving on the bound $K_2(12,2) \le 80$ [37]. That is, this is not a *proper* mixed code.

Theorem 26 $K_2(12,2) \le 78$.

Proof: Let $r = 11$ and $n = 12$. For $M = (11111111000)^T$, the following 39 codewords cover F_2^{11} with radius 2 using (**I**;**M**):

00001011110	01000000011	10001011011	11000010000
00001100000	01000010101	10001110100	11011101010
00010001000	01011101111	10010011111	11011111100
00010110110	01011111001	10010100101	11100010001
00101001111	01100000010	10010110011	11100101001
00101100011	01100101100	10101011000	11100111111
00101110101	01100111010	10101100110	11110010100
00110001011	01111000100	10110001110	11111000001
00110011101	01111010010	10110110000	11111010111
00110100111	10001001101	11000000110.	

Theorem 25 again gives that there are $39 \cdot 2 = 78$ codewords covering F_2^{12} with covering radius 2. Thus $K_2(12, 2) \leq 78$. □

If the mixed spaces $H = F_{q_1}^{n_1} F_{q_2}^{n_2} \cdots F_{q_m}^{n_m}$, and $G = F_{q_1}^{n'_1} F_{q_2}^{n'_2} \cdots F_{q_m}^{n'_m}$ have the property that $q_i \neq q_j$, when $i \neq j$, then A has the form displayed in Figure 7, and the number of nonequivalent matrices is

$$N = \prod_{i=1}^{i=m} \Phi_{q_i}(n'_i, n_i).$$

In the previous example, where we showed that $K_{3,2}(3, 7; 1) \leq 320$, $q_1 = 2$, $q_2 = 3$, $n_1 = 5$, $n'_1 = 7$, and $n_2 = n'_2 = 3$. Then $N = \Phi_2(7, 5) = 8$, so 8 different matrices have to be used in the attempts to find a covering with this method.

4 Conclusions

In this work construction methods for mixed covering codes are developed. Covering codes have the property that all words in a given space are within a fixed Hamming distance, called the covering radius, from some codeword. If the arities of the coordinates of a space vary, the space is called a mixed space and a code covering the space is analogously called a mixed covering code. The approach is very general considering any possible arities and combinations of these. As a matter of fact, this is the first general approach to the problem; earlier work has mainly concentrated on binary/ternary mixed codes, which are interesting due to possible applications to football pool systems, and perfect mixed codes.

The construction methods presented consist of generalizations of known constructions and completely new constructions. They are divided into three classes; direct constructions, constructions of new codes from old, and the matrix method. Most of the methods are combinatorial. In some of the constructions simulated annealing, a probabilistic optimization method, has turned out to be an indispensable aid. Through the constructions upper bounds for the minimal number of codewords in a code covering a fixed space with a given covering radius ($K_{q_1,q_2,\ldots,q_m}(n_1,n_2,\ldots,n_m;R)$) are improved. The matrix method is for example used to obtain a record-breaking binary code, proving $K_2(12,2) \leq 78$.

We develop constructions that can be generally applied, so a few very special methods are omitted due to their lack of generality. These are of two types; complicated combinatorial constructions, and constructions starting from near-coverings trying to cover the uncovered words in some way, using simulated annealing or some combinatorial method (cf. [71]). The near-coverings can be obtained for example by the general methods presented in this report.

In this report we give tables on upper bounds for codes of length 3 and 4, and arities $q_i \leq 5$. In connection to the presentation of the construction methods, examples of record-breaking codes are given. The most interesting upper bounds presented are $K_2(12,2) \leq 78(80)$, $K_{3,2}(3,7;1) \leq 320(332)$, and $K_5(5,3) \leq 9(10)$, where the old records are given in the parentheses. In addition to these new upper bounds, many interesting old upper bounds are explained. It has been our intention mainly to present the constructions here, and extensive tables on upper bounds for covering codes will be presented in future reports. Work is in progress to update the upper bounds given by Hämäläinen and Rankinen in [37] (It can be mentioned that some 40 of the upper bounds given by them have been improved since it was written). We are also going to present constructive methods for the bounds given by them since most of the references in that paper are to Finnish and Swedish football pool magazines, which are certainly not available to most researchers.

We hope that the results obtained and the notations introduced form a good starting-point for future research in the theory of mixed covering codes and other closely related areas within combinatorial mathematics. As a matter of fact, our work suggests a couple of research items. We conclude the report by mentioning these.

1. Investigate short mixed codes (especially of length 4). Try to find new general construction methods.
2. Calculate tables on bounds for mixed covering codes. Naturally, the parameters can be chosen in infinitely many ways, so the field is unlimited. (As mentioned it is our intention to publish an updated table on bounds for binary/ternary mixed codes.)
3. Calculate bounds for s-surjective mixed codes.
4. Calculate inequivalent matrices for the matrix method. Create "approximate" algorithms that are fast, but either do not give all inequivalent matrices or fail to see the equivalence in some cases.
5. Try to find other interesting applications of covering codes than those mentioned here.

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