

# Synthesizing Minimal Tile Sets for Patterned DNA Self-Assembly

Mika Göös and Pekka Orponen

Department of Information and Computer Science  
Aalto University School of Science and Technology (TKK)  
P.O. Box 15400, FI-00076 Aalto, Finland  
{mika.goos, pekka.orponen}@tkk.fi

**Abstract.** The Pattern self-Assembly Tile set Synthesis (PATS) problem is to determine a set of coloured tiles that self-assemble to implement a given rectangular colour pattern. We give an exhaustive branch-and-bound algorithm to find tile sets of minimum cardinality for the PATS problem. Our algorithm makes use of a search tree in the lattice of partitions of the ambient rectangular grid, and an efficient bounding function to prune this search tree. Empirical data on the performance of the algorithm shows that it compares favourably to previously presented heuristic solutions to the problem.

## 1 Introduction

### 1.1 Background

An appealing methodology for bottom-up manufacturing of nanoscale structures and devices is to use a self-assembling system of DNA tiles [11] to build a scaffold structure on which functional units are deposited [5, 9, 14]. A systematic approach to the design of self-assembling DNA scaffold structures was proposed and experimentally validated by Park et al. in [8]. However, as pointed out by Ma & Lombardi in [6], that design is wasteful of tile types, i.e. generally speaking the same scaffold structures can be assembled also from fewer types of specially-manufactured DNA complexes, thus reducing the requisite laboratory work.

Ma & Lombardi [6] formulated the task of minimizing the number of DNA tile types required to implement a given 2-D pattern abstractly as a combinatorial optimization problem, the **Patterned self-Assembly Tile set Synthesis** (PATS) problem, and proposed two greedy heuristics for solving it. In this paper, we present a systematic branch-and-bound approach to exploring the space of feasible PATS tilings, and assess its computational performance also experimentally. The method compares favourably to the heuristics proposed by Ma & Lombardi, finding noticeably smaller or even provably minimal tile sets in a reasonable amount of computation time. However, as the experimental results in Section 4 show, the computational problem still remains quite challenging for large patterns.

### 1.2 Overview

Our considerations take place in the **abstract Tile Assembly Model** (aTAM) of Winfree and Rothemund [10, 12, 13]. The key features of this model are as follows. The basic building blocks to construct different two-dimensional shapes are called **tiles**. A tile is a non-rotating unit square that has different kinds of **glues** of varying strengths associated with each of its four edges. Only finitely many different combinations of glues are allowed, that is, we consider only finitely many **glue types**; we have an unlimited supply of tiles, however. In the aTAM, tile assemblies are constructed through the process of self-assembly. Self-assembly begins with a **seed assembly**, an initial tile assembly, already in place in the discrete grid  $\mathbb{Z} \times \mathbb{Z}$ . A new tile can extend an existing assembly by binding itself into a suitable position in the grid: we require

a new tile to interact with existing tiles with sufficient strength to overcome a certain universal **temperature** threshold. A more detailed account of the aTAM model is given in section 2.1.

In the PATS problem [6], one associates a **colour** with each tile type and targets a specific coloured **pattern** within a rectangular assembly. The question is: given the desired colour pattern, what is the smallest set of (coloured) tile types that will self-assemble to implement it? The specifics of the PATS problem are given in section 2.2.

Our definition of the PATS problem restricts the self-assembly process to proceed in a uniform way. This simplification allows us to design efficient strategies for an exhaustive search. In section 3 we give full particulars of a novel branch-and-bound (B&B) algorithm for the PATS problem. For a pattern of size  $m \times n$ , we reduce the problem of finding a minimal tile set to the problem of finding a minimum-size **constructible** partition of  $[m] \times [n]$ . Here, constructibility of a partition can be verified in time polynomial in  $m$  and  $n$ . This leads us to construct a search tree in the lattice of partitions of the set  $[m] \times [n]$  and to find pruning strategies for this search tree. In the concluding sections 4 and 5 we give some performance data on the B&B algorithm and summarize our contributions.

## 2 Preliminaries

### 2.1 The abstract tile assembly model

Our notation is derived from those of [1, 4, 12]. First, to simplify our notations, let  $\mathcal{D} = \{N, E, S, W\}$  be the set of four functions  $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  corresponding to the cardinal directions (north, east, south, west) so that  $N(x, y) = (x, y+1)$ ,  $E(x, y) = (x+1, y)$ ,  $S = N^{-1}$  and  $W = E^{-1}$ .

Let  $\Sigma$  be a set of **glue types** and  $s : \Sigma \times \Sigma \rightarrow \mathbb{N}$  a **glue strength** function such that  $s(\sigma_1, \sigma_2) = s(\sigma_2, \sigma_1)$  for all  $\sigma_1, \sigma_2 \in \Sigma$ . In this paper, we only consider glue strength functions for which  $s(\sigma_1, \sigma_2) = 0$  if  $\sigma_1 \neq \sigma_2$ . A **tile type**  $t \in \Sigma^4$  is a quadruple  $(\sigma_N(t), \sigma_E(t), \sigma_S(t), \sigma_W(t))$  of glue types for each side of a unit square. Given a set  $\Sigma$  of glues, an **assembly**  $\mathcal{A}$  is a partial mapping from  $\mathbb{Z}^2$  to  $\Sigma^4$ . A **tile assembly system** (TAS)  $\mathcal{T} = (T, \mathcal{S}, s, \tau)$  consists of a finite set  $T$  of tile types, an assembly  $\mathcal{S}$  called the **seed assembly**, a glue strength function  $s$  and a **temperature**  $\tau \in \mathbb{Z}^+$  (we use  $\tau = 2$ ).

To formalize the self-assembly process, we first fix a TAS  $\mathcal{T} = (T, \mathcal{S}, s, \tau)$ . For two assemblies  $\mathcal{A}$  and  $\mathcal{A}'$  we write  $\mathcal{A} \rightarrow_{\mathcal{T}} \mathcal{A}'$  if there exists a pair  $(x, y) \in \mathbb{Z}^2$  and a tile  $t \in T$  such that  $\mathcal{A}' = \mathcal{A} \cup \{(x, y), t\}$ , where the union is disjoint, and

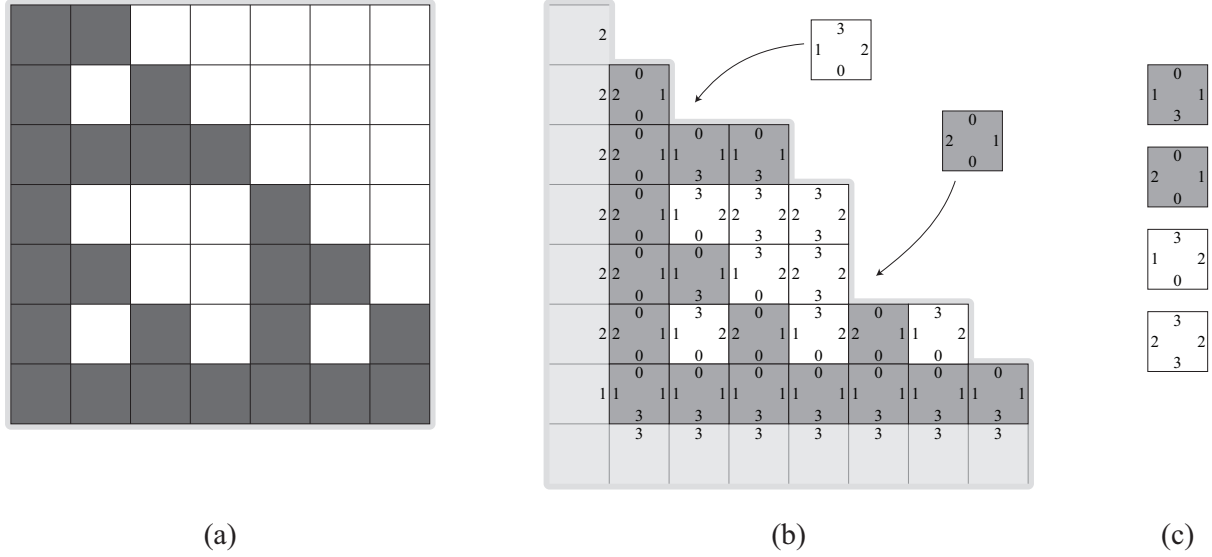
$$\sum_D s(\sigma_D(t), \sigma_{D^{-1}}(\mathcal{A}(D(x, y)))) \geq \tau , \quad (1)$$

where  $D$  ranges over those directions in  $\mathcal{D}$  for which  $\mathcal{A}(D(x, y))$  is defined. This is to say that a new tile can be adjoined to an assembly  $\mathcal{A}$  if the new tile shares a common boundary with tiles that bind it into place with total strength at least  $\tau$ .

Let  $\rightarrow_{\mathcal{T}}^*$  be the reflexive transitive closure of  $\rightarrow_{\mathcal{T}}$ . A TAS  $\mathcal{T}$  **produces** an assembly  $\mathcal{A}$  if  $\mathcal{A}$  is an **extension** of the seed assembly  $\mathcal{S}$ , that is if  $\mathcal{S} \rightarrow_{\mathcal{T}}^* \mathcal{A}$ . Let us denote by  $\text{Prod } \mathcal{T}$  the set of all assemblies produced by  $\mathcal{T}$ . This way, the pair  $(\text{Prod } \mathcal{T}, \rightarrow_{\mathcal{T}}^*)$  forms a partially ordered set. We say that a TAS  $\mathcal{T}$  is **deterministic** if for any assembly  $\mathcal{A} \in \text{Prod } \mathcal{T}$  and for every  $(x, y) \in \mathbb{Z}^2$  there exists at most one  $t \in T$  such that  $\mathcal{A}$  can be extended with  $t$  at position  $(x, y)$ . A TAS  $\mathcal{T}$  is deterministic precisely when  $\text{Prod } \mathcal{T}$  is a lattice. Also, the maximal elements in  $\text{Prod } \mathcal{T}$  are such assemblies  $\mathcal{A}$  that can not be further extended, that is, there do not exist assemblies  $\mathcal{A}'$  such that  $\mathcal{A} \rightarrow_{\mathcal{T}} \mathcal{A}'$ . These maximal elements are called **terminal assemblies**. We denote by  $\text{Term } \mathcal{T}$  the set of terminal assemblies of  $\mathcal{T}$ . If all **assembly sequences**

$$\mathcal{S} \rightarrow_{\mathcal{T}} \mathcal{A}_1 \rightarrow_{\mathcal{T}} \mathcal{A}_2 \rightarrow_{\mathcal{T}} \dots \quad (2)$$

terminate and  $\text{Term } \mathcal{T} = \{\mathcal{P}\}$  for some assembly  $\mathcal{P}$ , we say that  $\mathcal{T}$  **uniquely produces**  $\mathcal{P}$ .



**Fig. 1.** (a) A finite subset of the discrete Sierpinski triangle. This 2-colouring of the set  $[7] \times [7]$  defines an instance of the PATS problem. (b) Assembling the Sierpinski pattern with a TAS that has an appropriate seed assembly and a (coloured) tile set shown in (c).

## 2.2 The PATS problem

In this paper we restrict our attention to designing minimal tile assembly systems that construct a given pattern in a finite rectangular  $m$  by  $n$  grid  $[m] \times [n] \subseteq \mathbb{Z}^2$ . This problem was first discussed by Ma & Lombardi [6].

A mapping from  $[m] \times [n]$  onto  $[k]$  defines a  **$k$ -colouring** or a  **$k$ -coloured pattern**. To build a given pattern, we start with boundary tiles in place for the west and south borders of the  $m$  by  $n$  rectangle and keep extending this assembly by tiles with strength-1 glues.

**Definition 1 (Pattern self-Assembly Tile set Synthesis (PATS) [6]).**

**Given:** A  $k$ -colouring  $c: [m] \times [n] \rightarrow [k]$ .

**Find:** A tile assembly system  $\mathcal{T} = (T, \mathcal{S}, s, 2)$  such that

- P1. The tiles in  $T$  have bonding strength 1.
- P2. The domain of  $\mathcal{S}$  is  $[0, m] \times \{0\} \cup \{0\} \times [0, n]$  and all the terminal assemblies have the domain  $[0, m] \times [0, n]$ .
- P3. There exists a colouring  $d: T \rightarrow [k]$  such that for each terminal assembly  $\mathcal{A} \in \text{Term } \mathcal{T}$  we have  $d(\mathcal{A}(x, y)) = c(x, y)$  for all  $(x, y) \in [m] \times [n]$ .

In particular, we are interested in the **minimal solutions** (in terms of  $|T|$ ) to the PATS problem. By the same token, we can make the following assumption:

*Assumption 1.* In our TASs, every tile participates in assembling some terminal assembly.

Ma & Lombardi show a certain derivative of the above optimization problem NP-hard in [7]. However, to our knowledge, a proof of the NP-hardness of the PATS problem as stated above is lacking.

As an illustration, we construct a part of the Sierpinski triangle with a 4-tile TAS in Figure 1. We use natural numbers as glue labels in our figures.

In the literature, the seed assembly of a TAS is often taken to be a single seed tile [1, 12] whereas we consider an L-shaped seed assembly. These boundaries can always be self-assembled

using  $m + n + 1$  different tiles with strength-2 glues, but for practical purposes we allow for the possibility of using, for example, DNA origami techniques [3] to construct these boundary conditions.

Due to constraint *P1* the self-assembly process proceeds in a uniform manner directed from south-west to north-east. This paves the way for a simple characterization of deterministic TASs in the context of the PATS problem.

**Proposition 1.** *Solutions  $\mathcal{T} = (T, \mathcal{S}, s, 2)$  of the PATS problem are deterministic precisely when for each pair of glue types  $(\sigma_1, \sigma_2) \in \Sigma^2$  there is at most one tile type  $t \in T$  so that  $\sigma_S(t) = \sigma_1$  and  $\sigma_W(t) = \sigma_2$ .*

A simple observation reduces the work needed in finding minimal solutions of the PATS problem.

**Lemma 1.** *The minimal solutions of the PATS problem are deterministic TASs.*

*Proof.* For the sake of contradiction, suppose that  $\mathcal{N} = (T, \mathcal{S}, s, 2)$  is a minimal solution to a PATS problem instance and that  $\mathcal{N}$  is not deterministic. By the above proposition let tiles  $t_1, t_2 \in T$  be such that  $\sigma_S(t_1) = \sigma_S(t_2)$  and  $\sigma_W(t_1) = \sigma_W(t_2)$ . Consider the simplified TAS  $\mathcal{N}' = (T \setminus \{t_2\}, \mathcal{S}, s, 2)$ . We show that this, too, is a solution to the PATS problem, which violates the minimality of  $|T|$ .

Suppose  $\mathcal{A} \in \text{Term } \mathcal{N}'$ . If  $\mathcal{A} \notin \text{Term } \mathcal{N}$ , then some  $t \in T$  can be used to extend  $\mathcal{A}$  in  $\mathcal{N}$ . If  $t \in T \setminus \{t_2\}$ , then  $t$  could be used to extend  $\mathcal{A}$  in  $\mathcal{N}'$ , so we must have  $t = t_2$ . But since new tiles are always attached by binding to south and west sides of the tile,  $\mathcal{A}$  could then be extended by  $t_1$  in  $\mathcal{N}'$ . Thus, we conclude that  $\mathcal{A} \in \text{Term } \mathcal{N}$  and furthermore  $\text{Term } \mathcal{N}' \subseteq \text{Term } \mathcal{N}$ . This demonstrates that  $\mathcal{N}'$  has property *P2*. The properties *P1* and *P3* can be readily seen to hold for  $\mathcal{N}'$  as well. In terms of  $|T|$  we have found a more optimal solution—and a contradiction.  $\square$

*Assumption 2.* We consider only deterministic TASs in the sequel.

### 3 A branch-and-bound algorithm

We describe an exact algorithm to find minimal solutions to the PATS problem. We extend the methods of [6] to obtain an exhaustive branch-and-bound (B&B) algorithm. The idea of Ma & Lombardi [6] (following experimental work of [8]) is to start with an **initial tile set** that consists of  $m \cdot n$  different tiles, one for each of the grid positions in  $[m] \times [n]$ . Their algorithm then proceeds to merge tile types in order to minimize  $|T|$ . We formalize this search process as an exhaustive search in the set of all partitions of the set  $[m] \times [n]$ . In the following, we let a PATS instance be given by a fixed  $k$ -coloured pattern  $c : [m] \times [n] \rightarrow [k]$ .

#### 3.1 The search space

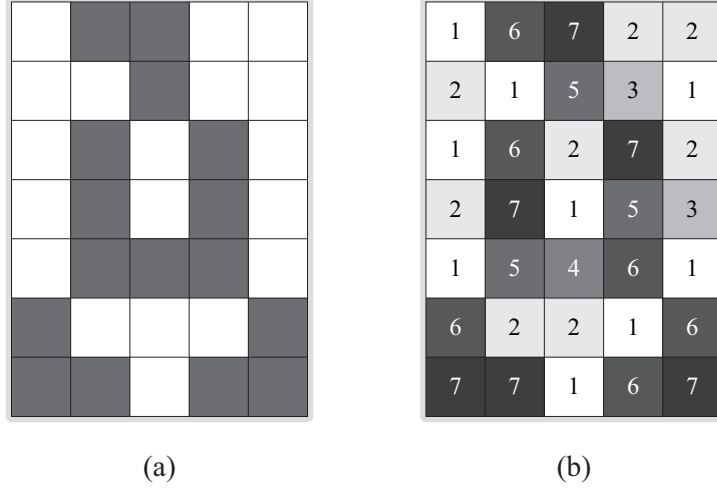
Let  $X$  be the set of partitions of the set  $[m] \times [n]$ . For partitions  $P, P' \in X$  we define a relation  $\sqsubseteq$  so that

$$P \sqsubseteq P' \iff \forall p' \in P' : \exists p \in P : p' \subseteq p . \quad (3)$$

Now,  $(X, \sqsubseteq)$  is a partially ordered set, and in fact, a lattice. If  $P \sqsubseteq P'$  we say that  $P'$  is a **refinement** of  $P$ , or that  $P$  is **coarser** than  $P'$ . Note that  $P \sqsubseteq P'$  implies  $|P| \leq |P'|$ .

The colouring  $c$  induces a partition  $P(c) = \{c^{-1}(\{i\}) \mid i \in [k]\}$  of the set  $[m] \times [n]$ . In addition, since every (deterministic) solution  $\mathcal{T} = (T, \mathcal{S}, s, 2)$  of the PATS problem uniquely produces some assembly  $\mathcal{A}$ , we associate with  $\mathcal{T}$  a partition  $P(\mathcal{T}) = \{\mathcal{A}^{-1}(\{t\}) \mid t \in \mathcal{A}([m] \times [n])\}$ . Here,  $|P(\mathcal{T})| = |T|$  due to our Assumptions 1 and 2. With this terminology, the condition *P3* in the definition of the PATS problem is equivalent to requiring that a TAS  $\mathcal{T}$  satisfies

$$P(c) \sqsubseteq P(\mathcal{T}) . \quad (4)$$



**Fig. 2.** (a) Partition  $A$ . (b) A partition  $M$  that is a refinement of  $A$  with  $|M| = 7$  parts.

We say that a partition  $P \in X$  is **constructible** if  $P = P(\mathcal{T})$  for some deterministic TAS  $\mathcal{T}$  with properties  $P1$  and  $P2$ . With this, we can rephrase our goal from the point of view of using partitions as the fundamental search space.

**Proposition 2.** *A minimal solution to the PATS problem corresponds to a partition  $P \in X$  such that  $P$  is constructible,  $P(c) \subseteq P$  and  $|P|$  is minimal.*

For example, the 2-coloured pattern in Figure 2a defines a 2-part partition,  $A$ , say. The 7-part partition  $M$  in Figure 2b is a refinement of  $A$  ( $A \subseteq M$ ) and in fact,  $M$  is constructible (see Figure 3b) and corresponds to a minimal solution of the PATS problem defined by the pattern  $A$ .

### 3.2 Determining constructibility

In this section we give an algorithm for deciding the constructibility of a given partition in polynomial time. To do this, we use the concept of most general (or least constraining) tile assignments. For simplicity, we assume the set of glue labels  $\Sigma$  to be infinite.

**Definition 2.** *Given a partition  $P$  of the set  $[m] \times [n]$ , a **most general tile assignment (MGTA)** is a function  $f : P \rightarrow \Sigma^4$  such that*

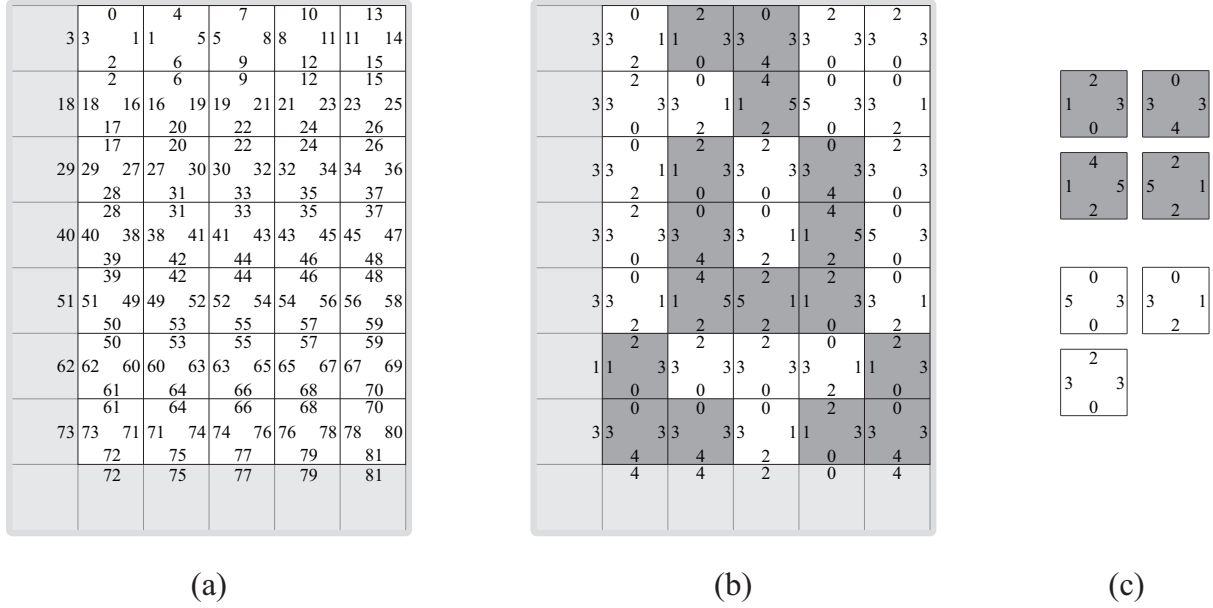
- A1. *When every position in  $[m] \times [n]$  is assigned a tile type according to  $f$ , any two adjacent positions agree on the glue type of the side between them.*
- A2. *For all assignments  $g : P \rightarrow \Sigma^4$  satisfying A1 we have<sup>1</sup>*

$$f(p_1)_{D_1} = f(p_2)_{D_2} \implies g(p_1)_{D_1} = g(p_2)_{D_2} \tag{5}$$

for all  $(p_1, D_1), (p_2, D_2) \in P \times \mathcal{D}$ .

To demonstrate this concept we present a most general tile assignment  $f : I \rightarrow \Sigma^4$  for the **initial partition**  $I = \{\{a\} \mid a \in [m] \times [n]\}$  in Figure 3a and a MGTA for the partition of Figure 2b in Figure 3b.

<sup>1</sup> To shorten the notation we write  $f(p)_D$  instead of  $\sigma_D(f(p))$ .



**Fig. 3.** (a) A MGTA for the constructible initial partition  $I$  (with a seed assembly in place). (b) Finished assembly for the pattern from Figure 2a. The tile set to construct this assembly is given in (c).

Given a partition  $P \in X$  and a function  $f : P \rightarrow \Sigma^4$ , we say that  $g : P \rightarrow \Sigma^4$  is obtained from  $f$  by **merging glues  $a$  and  $b$**  if for all  $(p, D) \in P \times \mathcal{D}$  we have

$$g(p)_D = \begin{cases} a, & \text{if } f(p)_D = b \\ f(p)_D, & \text{otherwise} \end{cases}. \quad (6)$$

A most general tile assignment for a partition  $P \in X$  can be found as follows. We start with a function  $f_0 : P \rightarrow \Sigma^4$  that assigns to each tile edge a unique glue type, or in other words, a function  $f_0$  so that the mapping  $(p, D) \mapsto f_0(p)_D$  is injective. Next, we go through all pairs of adjacent positions in  $[m] \times [n]$  (in some order) and require their matching sides to have the same glue type by merging the corresponding glues. This process generates a sequence of functions  $f_0, f_1, f_2, \dots, f_N = f$  and terminates after  $N \leq 2mn$  steps.

**Lemma 2.** *The above algorithm generates a most general tile assignment.*

*Proof.* By the end, we are left with a function  $f$  that satisfies property *A1* by construction. To see why property *A2* is satisfied, we again use the language of partitions.

Any assignment gives rise to a set of equivalence classes (or a partition) on  $P \times \mathcal{D}$ : Elements that are assigned the same glue type reside in the same equivalence class. The initial assignment  $f_0$  gives each part-direction pair a unique glue type, and thus, corresponds to the initial partition  $J = \{\{a\} \mid a \in P \times \mathcal{D}\}$ . In the algorithm, any glue merging operation corresponds to the combination of two equivalence classes.

The algorithm goes through a list of pairs  $\{\{a_i, b_i\}\}_{i=0}^{N-1}$  of elements from  $P \times \mathcal{D}$  that are required to have the same glue type. In this way, the list records necessary conditions for property *A1* to hold. This is to say that every assignment satisfying *A1* has to correspond to a partition that is coarser than each of the partitions in  $\mathcal{L} = \{J[a_i, b_i]\}_{i=0}^{N-1}$ , where  $J[a, b]$  is the partition obtained from the initial partition by combining parts  $a$  and  $b$ . Since the set  $(P \times \mathcal{D}, \sqsubseteq)$  is a lattice, there exists a unique greatest lower bound  $\inf \mathcal{L}$  of the partitions in  $\mathcal{L}$ . This is exactly the partition that the algorithm calculates in the form of the assignment  $f$ . As a greatest lower bound,  $\inf \mathcal{L}$  is finer than any partition corresponding to an assignment satisfying *A1*, but this is precisely the requirement for condition *A2*.  $\square$

The above analysis also gives the following.

**Corollary 1.** *For a given partition, MGTA's are unique up to relabeling of the glue types.*

Thus, for each partition  $P$ , we take **the MGTA for  $P$**  to be some canonical representative from the class of MGTA's for  $P$ .

For efficiency purposes, it is worth mentioning that MGTA's can be generated iteratively: A partition  $P \in X$  can be obtained by repeatedly combining parts starting from the initial partition  $I$ :

$$I = P_1 \sqsupseteq P_2 \sqsupseteq \dots \sqsupseteq P_N = P . \quad (7)$$

As a base case, a MGTA for  $I$  can be computed by the above algorithm. A MGTA for each  $P_{i+1}$  can be computed from a MGTA for the previous partition  $P_i$  by just a small modification: Let a MGTA  $f_i : P_i \rightarrow \Sigma^4$  be given for  $P_i$  and suppose  $P_{i+1}$  is obtained from  $P_i$  by combining parts  $p_1, p_2 \in P_i$ . Now, a MGTA  $f_{i+1}$  for  $P_{i+1}$  can be obtained from  $f_i$  by **merging tiles**  $f_i(p_1)$  and  $f_i(p_2)$ , that is, merging the glue types on the four corresponding sides.

We now give the conditions for a partition to be constructible in terms of MGTA's.

**Lemma 3.** *A partition  $P \in X$  is constructible iff the MGTA  $f : P \rightarrow \Sigma^4$  for  $P$  is injective and the tile set  $f(P)$  is deterministic in the sense of Proposition 1.*

*Proof.* “ $\Rightarrow$ ”: Let  $P \in X$  be constructible and let the MGTA  $f : P \rightarrow \Sigma^4$  for  $P$  be given. Let  $\mathcal{T}$  be a deterministic TAS such that  $P(\mathcal{T}) = P$ . The uniquely produced assembly of  $\mathcal{T}$  induces a tile assignment  $g : P \rightarrow \Sigma^4$  that satisfies property A1. Now using property A2 for the MGTA  $f$  we see that any violation of the injectivity of  $f$  or any violation of the determinism of the tile set  $f(P)$  would imply such violations for  $g$ . But since  $g$  corresponds to a constructible partition, no violations can occur for  $g$  and thus none for  $f$ .

“ $\Leftarrow$ ”: Let  $f : P \rightarrow \Sigma^4$  be an injective MGTA with deterministic tile set  $f(P)$ . Because  $f(P)$  is deterministic, we can choose glue types for a seed assembly  $\mathcal{S}$  so that the westernmost and southernmost tiles fall into place according to  $f$  in the self-assembly process. The TAS  $\mathcal{T} = (f(P), \mathcal{S}, s, 2)$ , with appropriate glue strengths  $s$ , then uniquely produces a terminal assembly that agrees with  $f$  on  $[m] \times [n]$ . This gives  $P(\mathcal{T}) \sqsubseteq P$ , but since  $f$  is injective,  $|P| = |f(P)| = |P(\mathcal{T})|$  and so  $P(\mathcal{T}) = P$ .  $\square$

### 3.3 An initial search DAG

Our algorithm performs an exhaustive search in the lattice  $(X, \sqsubseteq)$  searching for constructible partitions. In the search, we maintain and incrementally update MGTA's for every partition we visit. First, we describe simple branching rules to obtain a rooted directed acyclic graph search structure and later give rules to prune this DAG to a node-disjoint search tree.

The root of the DAG is taken to be the initial partition  $I$  that is always constructible. For each partition  $P \in X$  we next define the set  $C(P) \subseteq X$  of children of  $P$ . Our algorithm always proceeds by combining parts of the partition currently being visited, so for each  $P' \in C(P)$  we will have  $P' \sqsubseteq P$ . Say we visit a partition  $P \in X$ . We have two possibilities:

**C1.  $P$  is constructible:**

1. If  $P$  is not a refinement of the target pattern  $P(c)$ , that is if  $P(c) \not\sqsubseteq P$ , we can drop this branch of the search, since no possible descendant  $P' \sqsubseteq P$  can be a refinement of  $P(c)$  either. (i.e.  $C(P) = \emptyset$ )
2. In case  $P(c) \sqsubseteq P$ , we can use the MGTA for  $P$  to give a concrete solution to the PATS problem instance defined by the colouring  $c$ . To continue the search and to find more optimal solutions we consider each pair of parts  $\{p_1, p_2\} \subseteq P$  in turn and recursively visit the partition  $P[p_1, p_2]$  where the two parts are combined. In fact, by the above analysis, it is sufficient to consider only pairs of the same colour:

$$C(P) = \{P[p_1, p_2] \mid p_1, p_2 \in P, p_1 \neq p_2, \exists k \in P(c) : p_1, p_2 \subseteq k\} . \quad (8)$$

C2.  **$P$  is not constructible:** In this case the MGTA  $f$  for  $P$  gives  $f(p_1)_S = f(p_2)_S$  and  $f(p_1)_W = f(p_2)_W$  for some parts  $p_1 \neq p_2$ . We continue the search from partition  $P[p_1, p_2]$ :

$$C(P) = \{P[p_1, p_2]\} . \quad (9)$$

To guarantee that our algorithm finds the optimal solution in the case C2 above, we need the following.

**Lemma 4.** *Let  $P \in X$  be a non-constructible partition,  $f$  the MGTA for  $P$  and  $p_1, p_2 \in P$ ,  $p_1 \neq p_2$ , parts such that  $f(p_1)_S = f(p_2)_S$  and  $f(p_1)_W = f(p_2)_W$ . For all constructible  $C \sqsubseteq P$  we have  $C \sqsubseteq P[p_1, p_2]$ .*

*Proof.* Let  $P$ ,  $f$ ,  $p_1$  and  $p_2$  be as in the statement of the lemma. Let  $C \sqsubseteq P$  be a constructible partition and  $g : C \rightarrow \Sigma^4$  the MGTA for  $C$ . Since  $C$  is coarser than  $P$  we can obtain from  $g$  a tile assignment  $g' : P \rightarrow \Sigma^4$  such that  $g'(p) = g(q)$ , where  $q \in C$  is the unique part for which  $p \sqsubseteq q$ . The assignment  $g'$  has property A1 and so using A2 for the MGTA  $f$  we get that

$$f(p_1)_S = f(p_2)_S \ \& \ f(p_1)_W = f(p_2)_W \implies g'(p_1)_S = g'(p_2)_S \ \& \ g'(p_1)_W = g'(p_2)_W . \quad (10)$$

Now, since  $C$  is constructible, the identities  $g(q_1)_S = g(q_2)_S$  and  $g(q_1)_W = g(q_2)_W$  can not hold for any two different parts  $q_1, q_2 \in C$ . Looking at the definition of  $g'$ , we conclude that  $p_1 \sqsubseteq q$  and  $p_2 \sqsubseteq q$  for some  $q \in C$ . This demonstrates  $C \sqsubseteq P[p_1, p_2]$ .  $\square$

### 3.4 Pruning the DAG to a search tree

Computational resources should be saved by not visiting any partition twice. To keep the branches in our search structure node-disjoint, we maintain a list of graphs that store restrictions on the choices the search can make.

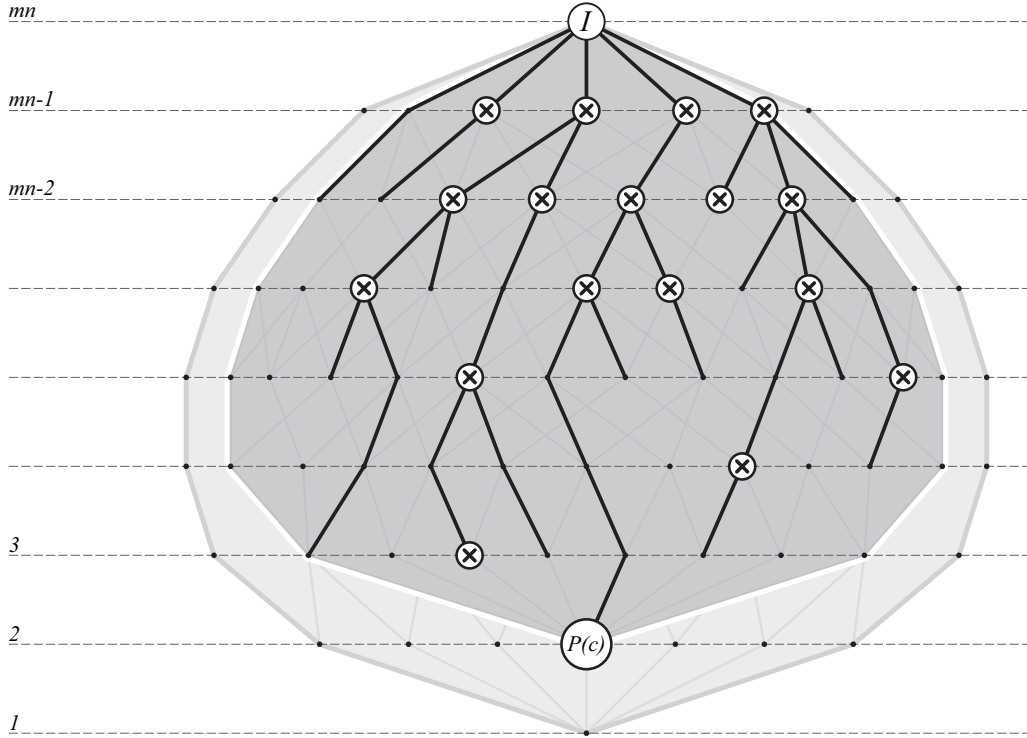
For each partition  $P \sqsupseteq P(c)$  we associate a family of undirected graphs  $\{G_k^P\}_{k \in P(c)}$ , one for each colour region of the pattern  $P(c)$ . Every part in  $P$  is represented by a vertex in the graph corresponding to the colour of the part. More formally, the vertex set  $V(G_k^P)$  is taken to be those parts  $p \in P$  for which  $p \subseteq k$ . (So now,  $\bigcup_{k \in P(c)} V(G_k^P) = P$ .) An edge  $\{p_1, p_2\} \in E(G_k^P)$  indicates that the parts  $p_1$  and  $p_2$  are not allowed ever to be combined in the search branch in question. When we start our search with the initial partition  $I$ , the edge sets are initially empty,  $E(G_k^I) = \emptyset$ . At each partition  $P$ , the graphs  $\{G_k^P\}_{k \in P(c)}$  have been determined inductively and the graphs for those children  $P' \in C(P)$  that we visit are defined as follows.

D1. **If  $P$  is constructible:** We choose some ordering  $\{p_i, q_i\}$ ,  $i = 1, \dots, N$ , for similarly coloured pairs of parts. Define  $l_i \in P(c)$ ,  $1 \leq i \leq N$  to be the colour of the pair  $\{p_i, q_i\}$ , so that  $p_i, q_i \subseteq l_i$ . Now, we visit a partition  $P[p_i, q_i]$  if and only if  $\{p_i, q_i\} \notin E(G_{l_i}^P)$ . When we decide to visit a child partition  $P' = P[p_j, q_j]$ , we define the edge sets  $\{E(G_k^{P'})\}_{k \in P(c)}$  as follows:

1. We start with the graphs  $\{G_k^P\}_{k \in P(c)}$  and add the edges  $\{p_i, q_i\}$  for all  $1 \leq i < j$  to their corresponding graphs. Call the resulting graphs  $\{G_k^*\}_{k \in P(c)}$ .
2. Finally, as we combine the parts  $p_j$  and  $q_j$  to obtain the partition  $P[p_j, q_j]$ , we merge the vertices  $p_j$  and  $q_j$  in the graph  $G_{l_j}^*$  (After merging, the neighbourhood of the new vertex  $p_j \cup q_j$  is the union of the neighbourhoods for  $p_j$  and  $q_j$  in  $G_{l_j}^*$ ). The graphs  $\{G_k^{P'}\}_{k \in P(c)}$  follow as a result.

D2. **If  $P$  is not constructible:** Here, the MGTA for  $P$  suggests a single child partition  $P' = P[p_1, p_2]$  for some  $p_1, p_2 \subseteq l \in P(c)$ . If  $\{p_1, p_2\} \in E(G_l^P)$ , we terminate this branch of the search. Otherwise, we define the graphs  $\{G_k^{P'}\}_{k \in P(c)}$  to be the graphs  $\{G_k^P\}_{k \in P(c)}$ , except that in  $G_l^{P'}$  the vertices  $p_1$  and  $p_2$  have to be merged.





**Fig. 4.** The search tree in the lattice  $(X, \subseteq)$ . We start with the initial partition  $I$  of size  $|I| = mn$ . The partition  $P(c)$  defines the PATS problem instance: We search for constructible partitions (drawn as crosses) in the sublattice (shaded with darker grey) consisting of those partitions that are refinements of  $P(c)$ . The search tree branches only at the constructible partitions and the tree branches are node-disjoint.

One can see that the outcome of this pruning process is a search tree that has node-disjoint branches and one in which every possible constructible partition is still guaranteed to be found. Figure 4 presents a sketch of the search tree.

Note that we are not usually interested in finding every constructible partition  $P \in X$ , but only in finding a minimal one (in terms of  $|P|$ ). Next, we give an efficient method to lower-bound the partition sizes of a given search branch.

### 3.5 The bounding function

Given a root  $P \in X$  of some subtree of the search tree, we ask: What is the smallest partition that can be found from this subtree? The nodes in the subtree rooted at  $P$  consists of those partitions  $P' \subseteq P$  that can be obtained from  $P$  by merging pairs of parts that are not forbidden by the graphs  $\{G_k^P\}_{k \in P(c)}$ . This merging process halts precisely when all the graphs  $\{G_k^{P'}\}_{k \in P(c)}$  have been reduced into cliques. As is well known, the size of the smallest clique that a graph  $G$  can be turned into by merging non-adjacent vertices is given by the **chromatic number**  $\chi(G)$  of the graph  $G$ . This immediately gives the following.

**Proposition 3.** *For every  $P' \subseteq P$  in the subtree rooted at  $P$  and constrained by  $\{G_k^P\}_{k \in P(c)}$ , we have*

$$\sum_{k \in P(c)} \chi(G_k^{P'}) \leq |P'| . \quad (11)$$

Determining the chromatic number of an arbitrary graph is an NP-hard problem. Fortunately, we can restrict our graphs to be of a special form: graphs that consist only of a clique and some isolated vertices. For these graphs, the chromatic numbers are given by the sizes of the cliques.

To see how to maintain graphs in this form, consider as a base case the initial partition  $I$ . Here,  $E(G_k^I) = \emptyset$  for all  $k \in P(c)$ , so  $G_k^I$  is of our special form—it has a clique of size 1. For a general partition  $P$ , we go through the branching rules D1-D2.

**D1:  $P$  is constructible:** Since we are allowed to choose an arbitrary ordering  $\{p_i, q_i\}$ ,  $i = 1, \dots, N$ , for the children  $P[p_i, q_i]$ , we design an ordering that preserves the special form of the graphs. For a graph  $G$  of our special form, let  $K(G) \subseteq V(G)$  consist of those vertices that are part of the clique in  $G$ . In the algorithm, we first set  $H_k = G_k^P$  for all  $k \in P(c)$  and repeat the following process until every graph  $H_k$  is a complete clique.

1. Pick some colour  $k \in P(c)$  and an isolated vertex  $v \in V(H_k) \setminus K(H_k)$ .
2. Process the pairs  $\{v, u\}$  for all  $u \in K(H_k)$  in some order. By the end, update  $H_k$  to include all the edges  $\{v, u\}$  that were just processed (the size of the clique in  $H_k$  increases by one).

A moment’s inspection reveals that when the graphs  $G_k^P$  are of our special form, so are all of the derived graphs passed on to the children of  $P$ .

**D2:  $P$  is not constructible:** If the algorithm decides to continue the search from a partition  $P' = P[p_1, p_2]$ , for some  $p_1, p_2 \subseteq l \in P(c)$ , we have  $\{p_1, p_2\} \notin E(G_l^P)$ . This means that either  $p_1, p_2 \in V(G_l^P) \setminus K(G_l^P)$ , in which case we are merging two isolated vertices, or one of  $p_1$  or  $p_2$  is part of the clique  $K(G_l^P)$ , in which case we merge an isolated vertex to the clique. In both cases, we maintain the special form in the graphs  $\{G_k^{P'}\}_{k \in P(c)}$ .

### 3.6 Traversing the search tree

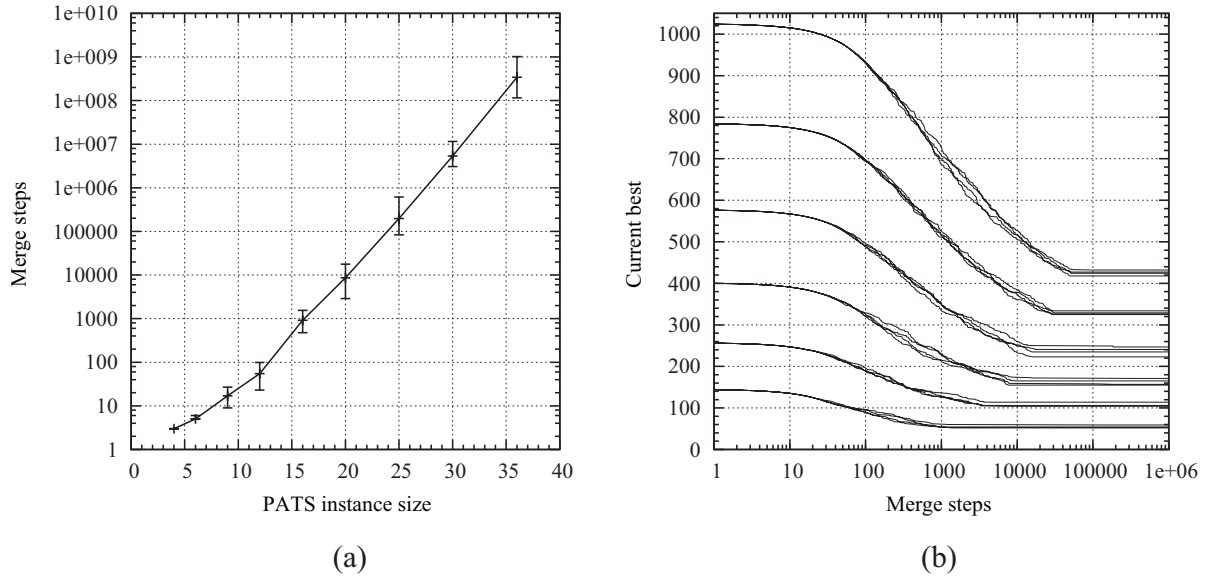
When running a B&B algorithm we maintain a “current best solution” discovered so far as a global variable. This solution gives an upper bound for the minimal value of the tile set size and can be used to prune such search branches that are guaranteed (by the bounding function) to only yield solutions worse than the current best. There are two general strategies to traverse a B&B search tree: **Depth-First Search** and **Best-First Search** [2]. Our description of the search tree for the lattice  $X$  is general enough to allow either of these strategies to be used in an actual implementation of the algorithm.

In the next section we give performance data on our DFS implementation of the B&B algorithm.

## 4 Results

The running time of our B&B algorithm is proportional—up to a polynomial factor—to the number of partitions the algorithm visits. Hence, we measure the running time in terms of the number of merge operations performed in the search. Figure 5a presents the running time of the algorithm to find a minimal solution for random 2-coloured instances of the PATS problem. The algorithm was executed for instance sizes  $2 \times 2, 2 \times 3, 3 \times 3, \dots, 5 \times 6$  and  $6 \times 6$ ; the 20th and 80th percentiles are shown alongside the median of 21 separate runs for each instance size. For the limiting case  $6 \times 6$ , the algorithm spent on the order of two hours of (median) computing time on a 2,61 GHz AMD processor.

Even though B&B search is an exact method, it can be used to find approximate solutions by running it for a suitable length of time. Figure 5b illustrates how the best solution found up to a point develops as increasingly many steps of the algorithm are run. The figure provides data on random 2-coloured instances of sizes from  $12 \times 12$  up to  $32 \times 32$ . Because we begin our search from the initial partition, the best solution at the first step is precisely equal to the instance size. For each size, several different patterns were used. The algorithm was cut off after  $10^6$  steps. By this time, an approximate reduction of 58% in the size of the tile set was achieved (cf. a reduction of 43.5% in [6]).



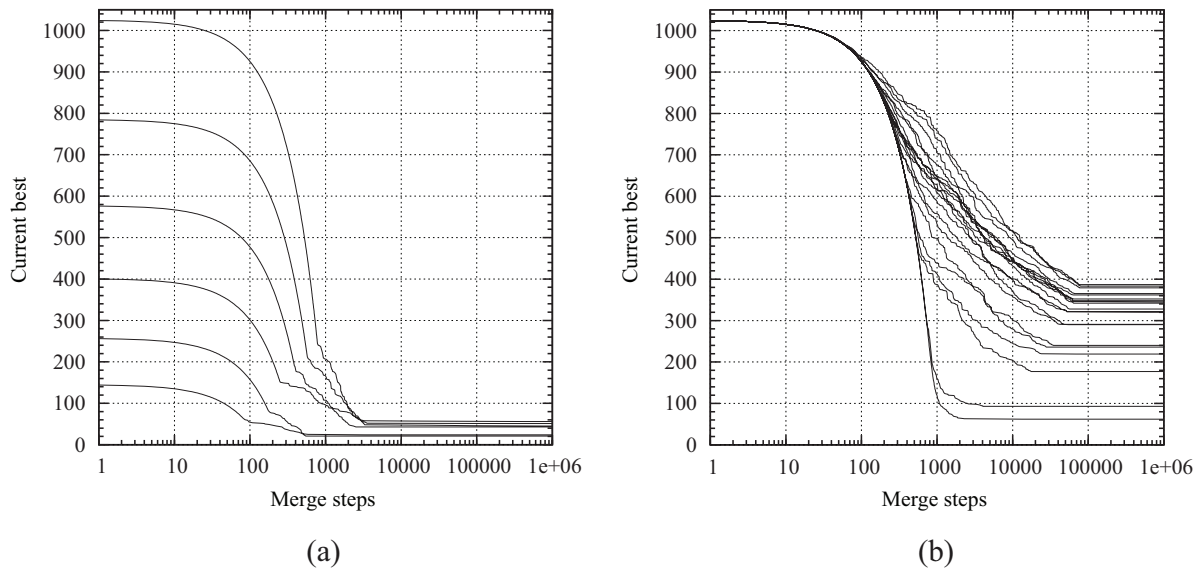
**Fig. 5.** (a) Running time of the algorithm (as measured by the number of merge operations) to solve random 2-coloured near-square-shaped instances of the PATS problem. (b) Evolution of the tile set size of the “current best solution” for several large random 2-coloured instances of the PATS problem.

Next, we consider two well known examples of structured patterns: the discrete Sierpinski triangle (part of which was shown in Figure 1) and the binary counter (see Figure 1 in [12]). A tile set of size 4 is optimal for both of these patterns. First, for the Sierpinski pattern, we get a tile reduction of well over 90% (cf. 45% in [6]) in Figure 6a. We used the same cutoff threshold and instance sizes as in Figure 5b. Our description of the B&B algorithm leaves some room for randomization in deciding which search branch a DFS is to explore next. This randomization does not seem to affect the search dramatically when considering the Sierpinski pattern—the separate single runs in Figure 6a are representative of an average randomized run. By contrast, for the binary counter pattern, randomized runs for single instance size do make a difference. Figure 6b depicts several separate runs for instance size  $32 \times 32$ . Here, each run brings about a reduction in solution size that oscillates between a reduction achieved on a random 2-coloured instance (5b) and a reduction achieved on the Sierpinski instance (6a). This suggests that, as is characteristic of DFS traversal, restarting the algorithm with a different random seed may help with large instances that have small optimal solutions.

## 5 Conclusion

We have presented an exact branch-and-bound algorithm for finding minimum-size tile sets that self-assemble a given  $k$ -coloured pattern in a uniform self-assembly setting. Simulation results indicate that our algorithm is able to find provably minimal tile sets for random instances of sizes up to  $6 \times 6$  and can give approximate solutions for larger instances as well.

One research direction to pursue would be to study tile sets that self-assemble an infinite, but finite-period pattern. Does this generalization reduce easily to the finite case? Do there exist minimal tile sets that tile the plane aperiodically while still producing a periodic colour pattern?



**Fig. 6.** Evolution of the “current best solution” for (a) the Sierpinski pattern and for (b) the binary counter pattern. Randomization in the DFS has a clear effect on the performance of the algorithm in the case of the binary counter pattern.

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